Description of the $2\nu\nu\beta\beta$ decay within a fully renormalized pnQRPA approach with a restored gauge symmetry

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(Dated: October 29, 2010)

Abstract

A many body Hamiltonian involving the mean field for a projected spherical single particle basis, the pairing interactions for alike nucleons, the dipole-dipole proton-neutron interactions in the particle-hole (ph) channel and the ph dipole pairing potential, is treated by the projected gauge of fully renormalized proton-neutron quasiparticle random phase approximation (PGFRpnQRPA) approach. The resulting wave functions and energies for the mother and the daughter nuclei are used to calculate the $2\nu\beta\beta$ decay rate and the process half life. For illustration, the formalism is applied for the decays ^{100}Mo$\rightarrow$ ^{100}Ru and ^{116}Cd$\rightarrow$ ^{116}Sn. The results are in good agreement with the corresponding experimental data. The Ikeda sum rule (ISR) is obeyed.

PACS numbers: 23.40.Hc,23.40.-s,21.10.Tg,21.60.Jz, 13.10,+q

I. INTRODUCTION

Double beta decay is one of the most exciting topic of nuclear physics because of the possible existence of neutrinoless $\beta\beta$ decay modes, which are very searched since they are connecting to the neutrino mass and neutrino nature. The calculation of the nuclear matrix elements for these decay modes is a challenge in the study of $\beta\beta$ decay process; however, this calculations implies the calculation not only of Gamow-Teller transitions but also of Fermi and tensor transitions.

The $2\nu\beta\beta$ process is interesting by its own but is also very attractive because it constitutes a test for some of the nuclear matrix elements (m.e.) which are used for the process of $0\nu\beta\beta$ decay. Discovery of this process may provide an answer for the fundamental question, whether neutrino is a Mayorana or a Dirac particle. The subject development is reflected be several review papers [1–7]. Our contribution described in this paper concerns the $2\nu\beta\beta$ process, which can be viewed as two consecutive and virtual single β^- decays. The formalism yielding closest results to the experimental data is the proton-neutron random phase approximation (pnQRPA) which includes the particle-hole (ph) and particle-particle (pp) [8] as independent two body interactions. The second leg of the $2\nu\beta\beta$ process is very sensitive to changing the relative strength of the later interaction, denoted hereafter by g_{pp}. It is worth mentioning that the two body interaction of ph type is repulsive while that of pp nature is attractive. Due to this feature there is a critical value for g_{pp} for which the first root of the pnQRPA equation vanishes. Actually, this is the signal that the $pnQRPA$ approach is no longer valid. Moreover, the g_{pp} value which corresponds to a transition amplitude which agrees with the corresponding experimental data is close to the mentioned critical value. That means that the result is not stable to adding corrections to the $pnQRPA$ picture. One improvement for the pnQRPA was achieved by one of us (AAR), in collaboration, in Refs.[9, 10], by using a boson expansion (BE) procedure. Another procedure of going beyond $pnQRPA$ is to renormalize the dipole two quasiparticle operators by replacing the scalar components of their commutators by their average values [11]. Such a renormalization is inconsistently achieved since the scattering operators are not renormalized. This lack of consistency was removed in Ref. [12, 13] where a fully renormalized pnQRPA (FRpnQRPA) is proposed.

Unfortunately, all higher pnQRPA procedures mentioned above have a common drawback

of violating the Ikeda sum rule (ISR) by an amount of about 20-30% [14]. It is believed that such a violation is caused by the gauge symmetry breaking. Consequently, a method of restoring this symmetry was formulated by the present authors in Ref. [15].

In this paper the results of Ref.[15] are improved in three respects: a) aiming at providing a unitary description of the process for the situations when the nuclei involved are spherical or deformed, here we use the projected spherical single particle basis defined in Ref.[16] and used for double beta decay in Refs.[17, 18]. b) the space of proton-neutron dipole configurations is split in three subspaces, one being associated to the single β^-, one to the β^+ process, and one spanned by the unphysical states. c) the correlations for the second leg of the process are mainly determined by the ph dipole-pairing term. A compact expression for the dispersion equation of energies is obtained from the linearized equations of motion of the basic transition operators corresponding to the two coupled processes. The numerical application is made for the $2\nu\beta\beta$ processes $^{100}Mo \rightarrow ^{100}Ru$ and ^{116}Cd$\rightarrow$ ^{116}Sn.

We shall describe the formalism and results according to the following plan. The single particle basis is briefly presented in Section II. The model Hamiltonian is given in Section III. The FRpnQRPA approach is discussed in Section IV, while the projected gauge of FRpnQRPA (PGFRpnQRPA) is the objective of Section V. The Gamow-Teller (GT) amplitude for the $2\nu\beta\beta$ process is given in Section VI. Numerical applications are shown in Section VII, while the final conclusions are drawn in Section VIII.

II. PROJECTED SINGLE PARTICLE BASIS

In Ref. [16], one of us (A.A.R.), and his collaborators, introduced an angular momentum projected single particle basis which seems to be appropriate for the description of the single particle motion in a deformed mean field generated by the particle-core interaction. This single particle basis has been used to study the collective M1 states in deformed nuclei [16, 19, 20] as well as the rate of double beta process [17, 21, 22].

In order to fix the necessary notations and moreover for the sake of a self-contained presentation, we describe briefly the main ideas underlying the construction of the projected single particle basis.

The single particle mean field is determined by a particle-core Hamiltonian:

$$\tilde{H} = H_{sm} + H_{core} - M\omega_0^2 r^2 \sum_{\lambda=0,2} \sum_{-\lambda\leq\mu\leq\lambda} \alpha^*_{\lambda\mu} Y_{\lambda\mu}. \tag{2.1}$$

where H_{sm} denotes the spherical shell model Hamiltonian, while H_{core} is a harmonic quadrupole boson ($b^+_{2\mu}$) Hamiltonian, associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2\mu}$.The monopole shape coordinate is to be determined from the volume conservation condition and thus expressed in terms of the quadrupole coordinates. These are related to the quadrupole boson operators by the canonical transformation:

$$\alpha_{2\mu} = \frac{1}{k\sqrt{2}}(b^\dagger_{2\mu} + (-)^\mu b_{2,-\mu}), \tag{2.2}$$

where k is an arbitrary C number.

Averaging $\tilde{H}$ on a given eigenstate of H_{sm}, denoted as usual by $|nljm\rangle$, one obtains a deformed quadrupole boson Hamiltonian whose eigenstate is an axially symmetric coherent state:

$$\Psi_g = exp[d(b^+_{20} - b_{20})]|0\rangle_b, \tag{2.3}$$

with $|0\rangle_b$ standing for the vacuum state of the boson operators and d a real parameter which simulates the nuclear deformation. On the other hand, averaging $\tilde{H}$ on Ψ_g one obtains a single particle mean field operator for the single particle motion, similar to the Nilsson Hamiltonian. Concluding, by averaging on a factor state of the particle core space the rotational symmetry is broken and the mean field mentioned above may generate, by diagonalization, a deformed basis for treating the many body interacting systems. However, this standard procedure is tedious since the final many body states should be projected over the angular momentum.

Our procedure defines first a spherical basis for the particle-core system, by projecting out the angular momentum from the deformed state

$$\Psi^{pc}_{nlj} = |nljm\rangle\Psi_g \tag{2.4}$$

One can prove that the subset of projected states :

$$\Phi^{IM}_{nlj}(d) = \mathcal{N}^I_{nlj} P^I_{MI}[|nljI\rangle\Psi_g] \equiv \mathcal{N}^I_{nlj}\Psi^{IM}_{nlj}(d). \tag{2.5}$$

forms an orthogonal basis for the particle-core system. The main properties of these projected spherical states have been presented in our previous works [17, 21, 22].

To the projected spherical states, one associates the 'deformed' single particle energies defined as the average values of the particle-core Hamiltonian $H' = \tilde{H} - H_{core}$.

$$\epsilon^I_{nlj} \;=\; \langle \Phi^{IM}_{nlj}(d)|H'|\Phi^{IM}_{nlj}(d)\rangle. \tag{2.6}$$

The deformation dependence of the new single particle energies is similar to that shown by the Nilsson model [23]. Therefore, the average values ϵ^I_{nlj} may be viewed as approximate expressions for the single particle energies in deformed Nilsson orbits [23]. We may account for the deviations from the exact eigenvalues by considering, at a later stage when a specific treatment of the many body system is performed, the exact matrix elements of the two body interaction.

As explained in Ref.[12], the redundancy problem concerning the number of degenerate states can be solved by changing the normalization of the model functions:

$$\langle \Phi^{IM}_{\alpha}|\Phi^{IM}_{\alpha}\rangle = 1 \Longrightarrow \sum_M \langle \Phi^{IM}_{\alpha}|\Phi^{IM}_{\alpha}\rangle = 2. \tag{2.7}$$

Due to this weighting factor the particle density function is providing the consistency result that the number of particles which can be distributed on the (2I+1) sub-states is at most 2, which agrees with the Nilsson model. Here α stands for the set of shell model quantum numbers nlj. Due to this normalization, the states Φ^{IM}_{α} used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2/(2I+1)}$.

The projected states might be thought of as eigenstates of an effective rotational invariant fermionic one-body Hamiltonian H_{eff}, with the corresponding energies given by Eq.(2.6).

$$H_{eff}\Phi^{IM}_{\alpha} = \epsilon^I_{\alpha}(d)\Phi^{IM}_{\alpha}. \tag{2.8}$$

As shown in Ref. [16] in the vibrational limit, $d \to 0$, the projected spherical basis goes to the spherical shell model basis and ϵ^I_{nlj} to the eigenvalues of H_{sm}.

III. THE MODEL HAMILTONIAN

We suppose that the states describing the nuclei involved in a $2\nu\beta\beta$ process are described by a many body Hamiltonian which may be written in the projected spherical basis as:

$$H = \sum_{\tau,\alpha,I,M} \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha}) c^{\dagger}_{\tau\alpha IM} c_{\tau\alpha IM} - \sum_{\tau,\alpha,I,I'} \frac{G_{\tau}}{4} P^{\dagger}_{\tau\alpha I} P_{\tau\alpha I'} + 2\chi \sum_{\substack{pn;p' \\ n';\mu}} \beta^{-}_{\mu}(pn)\beta^{+}_{-\mu}(p'n')(-)^{\mu}$$
$$-\chi_1 \sum_{\substack{pn;p' \\ n';\mu}} \left(\beta^{-}_{\mu}(pn)\beta^{-}_{-\mu}(p'n') + \beta^{+}_{-\mu}(p'n')\beta^{+}_{\mu}(pn)\right)(-1)^{1-\mu}, \tag{3.1}$$

where $c^{\dagger}_{\tau\alpha IM}(c_{\tau\alpha IM})$ denotes the creation (annihilation) operator of one nucleon of the type $\tau(=p,n)$ in the state Φ^{IM}_{α}, with α being an abbreviation for the set of quantum numbers nlj. In order to simplify the notations, hereafter the set of quantum numbers $\alpha(=nlj)$ will be omitted. The Hamiltonian H contains the mean field term, the pairing interaction for alike nucleons and the Gamow-Teller dipole-dipole interaction in the *ph* channel and the dipole *ph* pairing interaction. The corresponding strengths for the mentioned two-body interactions are denoted by $G_{\tau}(\tau = p,n), \chi, \chi_1$, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$P^{\dagger}_{\tau I} = \sum_{M} \frac{2}{2I+1} c^{\dagger}_{\tau IM} c^{\dagger}_{\widetilde{\tau IM}},$$
$$\beta^{-}_{\mu}(pn) = \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_{\mu}|nI'M'\rangle \frac{\sqrt{2}}{\hat{I}'} c^{\dagger}_{pIM} c_{nI'M'}. \tag{3.2}$$

The remaining operators from Eq.(3.1) can be obtained from the above defined operators, by Hermitian conjugation.

Passing to the quasiparticle (*qp*) representation through the Bogoliubov-Valatin transformation:

$$a^{\dagger}_{\tau IM} = U_{\tau I} c^{\dagger}_{\tau IM} - s_{IM} V_{\tau I} c_{\tau I-M}, s_{IM} = (-)^{I-M}, \;\; \tau = p,n, \;\; U^2_{\tau I} + V^2_{\tau I} = 1, \tag{3.3}$$

the first two terms of H are replaced by the independent quasiparticles term, $\sum E_{\tau I} a^{\dagger}_{\tau IM} a_{\tau IM}$, while the two-body dipole-dipole interactions are expressed in terms of the dipole two *qp* and the dipole *qp* density operators:

$$A^{\dagger}_{1\mu}(pn) = \sum C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^{\dagger}_{pI_pm_p} a^{\dagger}_{nI_nm_n}, \;\; A_{1\mu}(pn) = \left(A^{\dagger}_{1\mu}(pn)\right)^{\dagger},$$
$$B^{\dagger}_{1\mu}(pn) = \sum C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^{\dagger}_{pj_pm_p} a_{nI_nm_n}(-)^{I_n-m_n}, \;\; B_{1\mu}(pn) = \left(B^{\dagger}_{1\mu}(pn)\right)^{\dagger}. \tag{3.4}$$

IV. THE FULLY RENORMALIZED PNQRPA

In Ref.[12], we showed that all these operators can be renormalized as suggested by the commutation equations:

$$\begin{aligned}
\left[A_{1\mu}(k), A^{\dagger}_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[1 - \frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \\
\left[B^{\dagger}_{1\mu}(k), A^{\dagger}_{1\mu'}(k')\right] &\approx \left[B^{\dagger}_{1\mu}(k), A_{1\mu'}(k')\right] \approx 0, \\
\left[B_{1\mu}(k), B^{\dagger}_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[\frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right],\ k = (I_p, I_n).
\end{aligned} \tag{4.1}$$

Indeed, denoting by $C^{(1)}_{I_p,I_n}$ and $C^{(2)}_{I_p,I_n}$ the averages of the right hand sides of (4.1) with the renormalized RPA vacuum state, the renormalized operators defined as

$$\bar{A}_{1\mu}(k) = \frac{1}{\sqrt{C^{(1)}_k}} A_{1\mu},\ \bar{B}_{1\mu}(k) = \frac{1}{\sqrt{|C^{(2)}_k|}} B_{1\mu}, \tag{4.2}$$

obey boson like commutation relations:

$$\begin{aligned}
\left[\bar{A}_{1\mu}(k), \bar{A}^{\dagger}_{1\mu'}(k')\right] &= \delta_{k,k'}\delta_{\mu,\mu'}, \\
\left[\bar{B}_{1\mu}(k), \bar{B}^{\dagger}_{1\mu'}(k')\right] &= \delta_{k,k'}\delta_{\mu,\mu'} f_k,\quad f_k = sign(C^{(2)}_k).
\end{aligned} \tag{4.3}$$

Further, these operators are used to define the phonon operator:

$$C^{\dagger}_{1\mu} = \sum_k \left[X(k)\bar{A}^{\dagger}_{1\mu}(k) + Z(k)\bar{D}^{\dagger}_{1\mu}(k) - Y(k)\bar{A}_{1-\mu}(k)(-)^{1-\mu} - W(k)\bar{D}_{1-\mu}(k)(-)^{1-\mu}\right], \tag{4.4}$$

where $\bar{D}^{\dagger}_{1\mu}(k)$ is equal to $\bar{B}^{\dagger}_{1\mu'}(k')$ or $\bar{B}_{1\mu}(k)$ depending on whether f_k is + or -. The phonon amplitudes are determined by the equations:

$$\left[H, C^{\dagger}_{1\mu}\right] = \omega C^{\dagger}_{1\mu} \quad \left[C_{1\mu}, C^{\dagger}_{1\mu'}\right] = \delta_{\mu\mu'}. \tag{4.5}$$

Interesting properties for these equations and their solutions are discussed in our previous publications [12, 13].

V. THE PROJECTED GAUGE OF THE FULLY RENORMALIZED PNQRPA

The renormalized ground state, i.e., the vacuum state for the phonon operator defined by the FRpnQRPA approach, is a superposition of components describing the neighboring nuclei $(N-1,Z+1),(N+1,Z-1),(N+1Z+1),(N-1,Z-1)$. The first two components conserve the total number of nucleons (N+Z) but violates the third component of isospin, T_3. By contrast, the last two components violates the total number of nucleons but preserve T_3. Actually, the last two components contribute to the violation of the ISR. However, one can construct linear combinations of the basic operators $A^\dagger, A, B^\dagger, B$ which excite the nucleus (N,Z) to the nuclei $(N-1,Z+1),(N+1,Z-1),(N+1,Z+1),(N-1,Z-1)$, respectively. These operators are:

$$\begin{aligned}
\mathcal{A}^\dagger_{1\mu}(pn) &= U_pV_nA^\dagger_{1\mu}(pn)+U_nV_pA_{1,-\mu}(pn)(-)^{1-\mu}+U_pU_nB^\dagger_{1\mu}(pn)-V_pV_nB_{1,-\mu}(pn)(-)^{1-\mu},\\
\mathcal{A}_{1\mu}(pn) &= U_pV_nA_{1\mu}(pn)+U_nV_pA^\dagger_{1,-\mu}(pn)(-)^{1-\mu}+U_pU_nB_{1\mu}(pn)-V_pV_nB^\dagger_{1,-\mu}(pn)(-)^{1-\mu},\\
\mathbf{A}^\dagger_{1\mu}(pn) &= U_pU_nA^\dagger_{1\mu}(pn)-V_pV_nA_{1,-\mu}(pn)(-)^{1-\mu}-U_pV_nB^\dagger_{1\mu}(pn)-V_pU_nB_{1,-\mu}(pn)(-)^{1-\mu},\\
\mathbf{A}_{1\mu}(pn) &= U_pU_nA_{1\mu}(pn)-V_pV_nA^\dagger_{1,-\mu}(pn)(-)^{1-\mu}-U_pV_nB_{1\mu}(pn)-V_pU_nB^\dagger_{1,-\mu}(pn)(-)^{1-\mu}.
\end{aligned}$$

Indeed, in the particle representation these operators have the expressions:

$$\mathcal{A}^\dagger_{1\mu}(pn) = -\left[c^\dagger_p c^{\sim}_{\overline{n}}\right]_{1\mu},\ \ \mathcal{A}_{1\mu}(pn) = -\left[c^\dagger_p c^{\sim}_{\overline{n}}\right]^\dagger_{1\mu}, \tag{5.1}$$

$$\mathbf{A}^\dagger_{1\mu}(pn) = \left[c^\dagger_p c^\dagger_n\right]_{1\mu},\ \ \mathbf{A}_{1\mu}(pn) = \left[c^\dagger_p c^\dagger_n\right]^\dagger_{1\mu}. \tag{5.2}$$

In terms of the new operators the many body Hamiltonian is:

$$\begin{aligned}
H &= \sum_{\tau jm} E_{\tau j}a^\dagger_{\tau jm}a_{\tau jm}+2\chi\sum_{pn,p'n';\mu}\sigma_{pn;p'n'}\mathcal{A}^\dagger_{1\mu}(pn)\mathcal{A}_{1\mu}(p'n')-\chi_1\sum_{\substack{pn;p'\\ n';\mu}}\sigma_{pn;p'n'}\\
&\times\ \left(\mathcal{A}^\dagger_{1\mu}(pn)\mathcal{A}^\dagger_{1,-\mu}(p'n')+\mathcal{A}_{1,-\mu}(p'n')\mathcal{A}_{1\mu}(pn)\right)(-)^{1-\mu},\\
&\sigma_{pn;p'n'} = \frac{2}{3\hat{I}_n\hat{I}_{n'}}\langle I_p||\sigma||I_n\rangle\langle I_{p'}||\sigma||I_{n'}\rangle.
\end{aligned} \tag{5.3}$$

Here $E_{\tau I}$ denotes the quasiparticle energy. If instead of the dipole ph pairing interaction we consider the proton-neutron two-body pp interaction, then the third term of H would be:

$$H_3 = -2\chi_1\sum_{\substack{pn;p'\\ n';\mu}}\sigma_{pn;p'n'}\mathbf{A}^\dagger_{1\mu}(pn)\mathbf{A}_{1\mu}(p'n'). \tag{5.4}$$

Since we are interested in describing the harmonic modes which preserve the total number of nucleons, the term H_3 would not contribute at the RPA level. Indeed, this term defines a deuteron type excitation, and consequently modifies the total number of nucleons.

The equations of motion of the operators defining the phonon operator are determined by the commutation relations:

$$\left[\mathcal{A}_{1\mu}(pn), \mathcal{A}^{\dagger}_{1\mu'}(p'n')\right] \approx \delta_{\mu,\mu'}\delta_{j_p,j_{p'}}\delta_{j_n,j_{n'}}\left[U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{\hat{j}_n^2}\hat{N}_n - \frac{U_p^2 - V_p^2}{\hat{j}_p^2}\hat{N}_p\right]. \quad (5.5)$$

The average of the r.h. side of this equation with the $PGFRpnQRPA$ vacuum state is denoted by:

$$D_1(pn) = U_p^2 - U_n^2 + \frac{1}{2I_n+1}(U_n^2 - V_n^2)\langle\hat{N}_n\rangle - \frac{1}{2I_p+1}(U_p^2 - V_p^2)\langle\hat{N}_p\rangle. \quad (5.6)$$

The equations of motion show that the two qp energies are renormalized too:

$$E^{ren}(pn) = E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2). \quad (5.7)$$

The space of pn dipole states, $\mathcal{S}$, is written as a sum of three subspaces defined as:

$$\begin{aligned}
\mathcal{S}_+ &= \{(p,n)|D_1(pn) > 0, \;\; E^{ren}(pn) > 0,\}\,, \;\; \mathcal{S}_- = \{(p,n)|D_1(pn) < 0, \;\; E^{ren}(pn) < 0,\}\,, \\
\mathcal{S}_{sp} &= \mathcal{S} - (\mathcal{S}_+ + \mathcal{S}_-)\,, \\
\mathcal{N}_\pm &= \dim(\mathcal{S}_\pm), \;\; \mathcal{N}_{sp} = \dim(\mathcal{S}_{sp}), \\
\mathcal{N} &= \mathcal{N}_+ + \mathcal{N}_- + \mathcal{N}_{sp}.
\end{aligned} \quad (5.8)$$

The third line of the above equations specify the dimensions of these subspaces. In $\mathcal{S}_+$ one defines the renormalized operators:

$$\bar{\mathcal{A}}^{\dagger}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}^{\dagger}_{1\mu}(pn), \;\; \bar{\mathcal{A}}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}_{1\mu}(pn), \quad (5.9)$$

while in $\mathcal{S}_-$ the renormalized operators are:

$$\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}_{1\mu}(pn), \;\; \bar{\mathcal{F}}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}^{\dagger}_{1\mu}(pn). \quad (5.10)$$

Indeed, the operator pairs $\mathcal{A}_{1\mu}, \mathcal{A}^{\dagger}_{1\mu}$ and $\mathcal{F}_{1\mu}, \mathcal{F}^{\dagger}_{1\mu}$ satisfy commutation relations of boson type. An RPA treatment within $\mathcal{S}_{sp}$ would yield either vanishing or negative energies. The corresponding states are therefore spurious.

The equations of motion for the renormalized operators read:

$$\begin{aligned}
\Big[H,\bar{\mathcal{A}}^{\dagger}_{1\mu}(pn)\Big] &= E^{ren}(pn)\bar{\mathcal{A}}^{\dagger}_{1\mu}(pn) + 2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1\mu}(p_1n_1) - 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1\mu}(p_1n_1)\\
&+ 2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1-\mu}(-1)^{1-\mu}(p_1n_1) - 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1n_1),\\
\Big[H,\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn)\Big] &= |E^{ren}(pn)|\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn) + 2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1\mu}(p_1n_1) - 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1\mu}(p_1n_1)\\
&+ 2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1n_1) - 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1-\mu}(p_1n_1)(-1)^{1-\mu},\\
\Big[H,\bar{\mathcal{A}}_{1\mu}(pn)\Big] &= -E^{ren}(pn)\bar{\mathcal{A}}_{1\mu}(pn) - 2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1\mu}(p_1n_1) + 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1\mu}(p_1n_1)\\
&- 2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1,-\mu}(p_1n_1)(-1)^{1-\mu} + 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1-\mu}(p_1n_1)(-1)^{1-\mu},\\
\Big[H,\bar{\mathcal{F}}_{1\mu}(pn)\Big] &= -|E^{ren}(pn)|\bar{\mathcal{F}}_{1\mu}(pn) - 2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1\mu}(p_1n_1) + 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1\mu}(p_1n_1)\\
&- 2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1,-\mu}(p_1n_1)(-1)^{1-\mu} + 2\chi_1\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1-\mu}(p_1n_1)(-1)^{1-\mu},
\end{aligned} \tag{5.11}$$

where:

$$\sigma^{(1)}_{pn;p_1n_1} = \frac{2}{\hat{1}\hat{I}_n}\langle p||\sigma||n\rangle|D_1(pn)|^{1/2}\frac{2}{\hat{1}\hat{I}_{n_1}}\langle p_1||\sigma||n_1\rangle|D_1(p_1n_1)|^{1/2} \equiv T_{pn}T_{p_1n_1}. \tag{5.12}$$

The phonon operator is defined as:

$$\Gamma^{\dagger}_{1\mu} = \sum_{k=(pn)}\Big[X(k)\bar{\mathcal{A}}^{\dagger}_{1\mu}(k) + Z(k)\bar{\mathcal{F}}^{\dagger}_{1\mu}(k) - Y(k)\bar{\mathcal{A}}_{1-\mu}(k)(-)^{1-\mu} - W(k)\bar{\mathcal{F}}_{1-\mu}(k)(-)^{1-\mu}\Big]. \tag{5.13}$$

with the amplitudes determined by the equations:

$$\Big[H,\Gamma^{\dagger}_{1\mu}\Big] = \omega\Gamma^{\dagger}_{1\mu},\ \Big[\Gamma_{1\mu},\Gamma^{\dagger}_{1\mu'}\Big] = \delta_{\mu,\mu'}. \tag{5.14}$$

Thus the phonon amplitudes are obtained by solving the PGFRpnQRPA equations:

$$\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12}\\ A_{21} & A_{22} & B_{21} & B_{22}\\ -B_{11} & -B_{12} & -A_{11} & -A_{12}\\ -B_{21} & -B_{22} & -A_{21} & -A_{22}\end{pmatrix}\begin{pmatrix} X(pn)\\ Z(pn)\\ Y(pn)\\ W(pn)\end{pmatrix} = \omega\begin{pmatrix} X(p_1n_1)\\ Z(p_1n_1)\\ Y(p_1n_1)\\ W(p_1n_1)\end{pmatrix}. \tag{5.15}$$

where the following notations have been used:

$$(A_{11}) = E^{ren}(pn)\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},$$

$$
\begin{aligned}
(A_{12}) &= -2\chi_1\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{11}) &= -2\chi_1\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{12}) &= 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(A_{21}) &= -2\chi_1\sigma^{(1)T}_{p_1n_1;pn},\\
(A_{22}) &= |E^{ren}(pn)|\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{21}) &= 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{22}) &= -2\chi_1\sigma^{(1)T}_{p_1n_1;pn}.
\end{aligned}
\tag{5.16}
$$

Here the index T suggests the fact that the matrix is transposed. Matrix dimension for A_{11} and B_{11} is $\mathcal{N}_+ \times \mathcal{N}_+$ while for A_{22} and B_{22} is $\mathcal{N}_- \times \mathcal{N}_-$. The off diagonal sub-matrices A_{12} and B_{12} have the dimension $\mathcal{N}_+ \times \mathcal{N}_-$ while A_{12} and B_{12} are of the $\mathcal{N}_- \times \mathcal{N}_+$ type.

In order to solve Eqs.(5.15) we need to know $D_1(pn)$ and, therefore, the averages of the qp's number operators, $\hat{N}_p$ and $\hat{N}_n$. These are written first in particle representation and then the particle number conserving term is expressed as a linear combination of $\mathcal{A}^\dagger\mathcal{A}$ and $\mathcal{F}^\dagger\mathcal{F}$ chosen such that their commutators with $\mathcal{A}^\dagger, \mathcal{A}$ and $\mathcal{F}^\dagger, \mathcal{F}$ are preserved. The final result is:

$$
\begin{aligned}
\langle \hat{N}_p \rangle &= V_p^2(2I_p+1) + 3(U_p^2 - V_p^2)\Big(\sum_{\substack{n',k\\(p,n')\in\mathcal{S}_+}} D_1(p,n')(Y_k(p,n'))^2 - \sum_{\substack{n',k\\(p,n')\in\mathcal{S}_-}} D_1(p,n')(W_k(p,n'))^2\Big),\\
\langle \hat{N}_n \rangle &= V_n^2(2I_n+1) + 3(U_n^2 - V_n^2)\Big(\sum_{\substack{p',k\\(p',n)\in\mathcal{S}_+}} D_1(p',n)(Y_k(p',n))^2 - \sum_{\substack{p',k\\(p',n)\in\mathcal{S}_-}} D_1(p',n)(W_k(p',n))^2\Big).
\end{aligned}
\tag{5.17}
$$

Eqs. (5.15), (5.17) and (5.6) are to be simultaneously considered and solved iteratively. It is worth mentioning that using the quasiparticle representation for the basic operators $\mathcal{A}^\dagger_{1\mu}, \mathcal{F}^\dagger_{1\mu}, \mathcal{A}_{1,-\mu}(-1)^{1-\mu}, \mathcal{F}_{1,-\mu}(-)^{1-\mu}$, one obtains for $\Gamma^\dagger_{1\mu}$ an expression which involves the scattering pn operators. Thus, the present approach is, indeed, the $PGFRpnQRPA$.

It is worth noting that the compatibility condition for the $PGpnQRPA$ equations (5.15) can be written in a compact form as a dispersion equation for the excitation energies. This is explicitly given in Appendix A.

VI. THE $2\nu\beta\beta$ PROCESS

The formalism presented above was used to describe the $2\nu\beta\beta$ process. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q-value of the double beta decay process

$$\Delta E = m_e c^2 + \frac{1}{2} Q_{\beta\beta}, \tag{6.1}$$

the reciprocal value of the $2\nu\beta\beta$ half life can be factorized as:

$$(T_{1/2}^{2\nu\beta\beta})^{-1} = F |M_{GT}(0_i^+ \rightarrow 0_f^+)|^2, \tag{6.2}$$

where F is an integral on the phase space, independent of the nuclear structure, while M_{GT} stands for the Gamow-Teller transition amplitude and has the expression :

$$M_{GT} = \sqrt{3} \sum_{k,k'} \frac{{}_i\langle 0||\beta_i^+||1_k\rangle_{ii}\langle 1_k|1_{k'}\rangle_{ff}\langle 1_{k'}||\beta_f^+||0\rangle_f}{E_k + \Delta E + E_{1^+}}. \tag{6.3}$$

In the above equation, the denominator consists of three terms: a) ΔE, which was already defined, b) the average value of the k-th $PDFRpnQRPA$ energy normalized to the particular value corresponding to k=1, and c) the experimental energy for the lowest 1^+ state. The indices carried by the β^+ operators indicate that they act in the space spanned by the $PGFRpnQRPA$ states associated to the initial (i) or final (f) nucleus, respectively. The overlap m.e. of the single phonon states in the initial and final nuclei respectively, are calculated within $PGFRpnQRPA$. In Eq.(6.3), the Rose convention for the reduced m.e. is used [24].

Note that if we restrict the pn space to $\mathcal{S}_+$ and moreover the χ_1 interaction is missing, M_{GT} vanishes due to the second leg of the transition. Indeed, the m.e. associated to the daughter nucleus is of the type ${}_f\langle 0|(c_n^\dagger c_p)_{1\mu}(c_n^\dagger c_p)_{1\mu}|o\rangle_f$, which is equal to zero due to the Pauli principle restriction. In this case the equations of motion are of Tam Dankoff type and therefore the ground state correlations are missing. In order to induce the necessary correlations we have either to extend the formalism in the space $\mathcal{S}_-$, or to allow the ph excitations to interact via a pairing like force. Here the two effects are simultaneously considered. Also, we remark that the operator $\bar{\mathcal{A}}_{1\mu}^\dagger$ plays the role of a β^- transition operator, while when $\bar{\mathcal{F}}_{1\mu}^\dagger$ is applied on the ground state of the daughter nucleus, it induces a β^+ transition. Therefore, the $\beta\beta$ decay cannot be described by considering the β^- transition alone.

	d	k	G_p[MeV]	G_n[MeV]	ISR	logft	χ [MeV]	χ_1[MeV]
^{100}Mo	-1.5	5.5	0.18	0.288	15.995	$^{100}\mathrm{Mo}\overset{\beta^+/EC}{\leftarrow}{}^{100}\mathrm{Tc}$	0.232	1.406
						$4.45^{+0.18}_{-0.30}$ 4.65		
^{100}Ru	-0.6	5.5	0.15	0.255	12.002	$^{100}\mathrm{Tc}\overset{\beta^-}{\rightarrow}{}^{100}\mathrm{Ru}$	0.232	1.406
						4.66 4.12		
^{116}Cd	-1.8	12.	0.15	0.282	20.07	$^{116}\mathrm{Cd}\overset{\beta^+/EC}{\leftarrow}{}^{116}\mathrm{In}$	0.2	1.308
						$4.39^{+0.1}_{-0.15}$ 4.29		
^{116}Sn	-1.2	12.	0.12	0.2458	16.007	$^{116}\mathrm{In}\overset{\beta^-}{\rightarrow}{}^{116}\mathrm{Sn}$	0.2	1.308
						4.662 4.08		

TABLE I: The deformation parameter d, the pairing interaction strengths for protons (G_p) and and neutrons (G_n) and the GT dipole interaction χ used in our calculations. We also give the parameter k relating the quadrupole coordinates and bosons (this is involved in the expression of the single particle energies) as well as the resulting logft values characterizing the β^+/EC and β_- transitions of ^{100}Tc and ^{116}In, respectively. The results for logft values, given in the right column, are compared to the experimental data from the left column.

VII. NUMERICAL APPLICATION

For illustration, we present the results for the transitions ^{100}Mo$\rightarrow{}^{100}$Ru and ^{116}Cd$\rightarrow{}^{116}$Sn . For these cases the energy corrections involved in Eq.(6.3) are:

$$\Delta E(^{100}Mo) = 2.026MeV, \quad E_{1^+}(^{100}Tc) = 0.0MeV.$$
$$\Delta E(^{116}Cd) = 1.916MeV, \quad E_{1^+}(^{116}In) = 0.0MeV. \tag{7.1}$$

The parameters defining the single particle energies are those of the spherical shell model, the deformation parameter d and the parameter k relating the quadrupole coordinate with the quadrupole bosons as shown in Eq.(2.2).

These are fixed as described in Ref.[18]. The core system for the two decays is defined by $(Z, N) = (20, 20)$. Labeling the states according to their energies ordering, the single particle space is defined by the indices interval $[11, 55]$ and $[11, 65]$, respectively. The dimensions for the spaces $(\mathcal{S}_+, \mathcal{S}_-, \mathcal{S})$ are $(137, 1, 163)$ and $(139, 2, 175)$ for ^{100}Mo and ^{100}Ru , while for the mother and daughter nuclei of the decay ^{116}Cd$\rightarrow{}^{116}$Sn, they are $(189, 0, 219)$ and $(182, 4, 219)$, respectively. For the both processes considered here eight iterations were

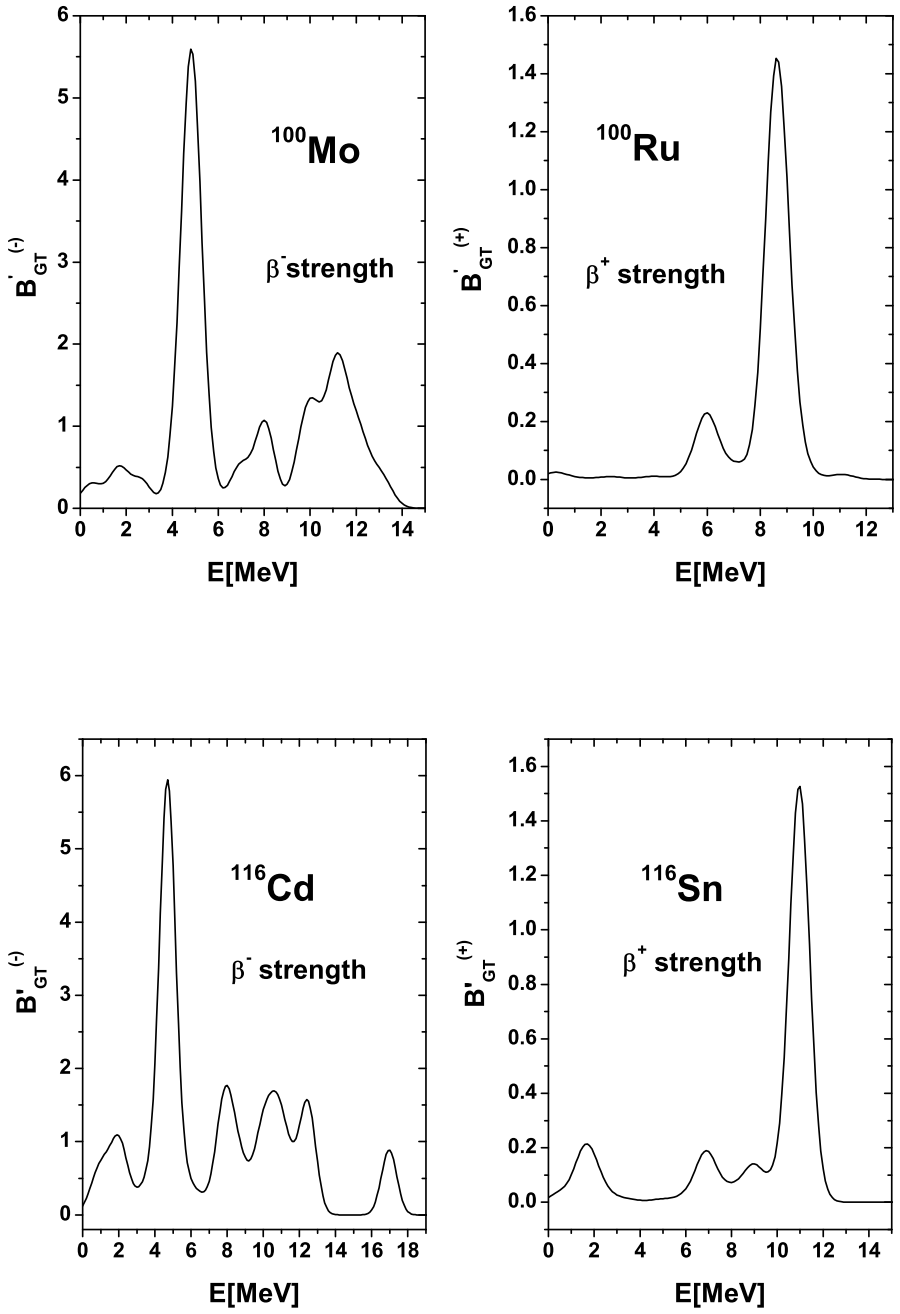

FIG. 1: One third of the single β^- strength for the mother nucleus, ^{100}Mo (upper-left panel) and ^{116}Cd (bottom-left), and one third of the β^+ strength for the daughter nucleus, ^{100}Ru (upper-right panel) and ^{116}Sn (bottom-right panel), folded by a Gaussian function with a width of 1 MeV, are plotted as functions of the corresponding energies yielded by the present formalism. Note that for a given nucleus the difference $B'^{(-)}_{GT} - B'^{(+)}_{GT}$ is to be compared with the reduced ISR value i.e.,$N - Z$.

necessary in order that the iteration process reaches the convergence. The strength of the dipole *pn* two-body interaction is usually taken to be

$$\chi = \frac{5.2}{A^{0.7}} MeV. \tag{7.2}$$

This expression was obtained by fitting the positions of the GT resonances in ^{40}Ca, ^{90}Zr and ^{208}Pb [25]. This expression provides for χ the values 0.207 and 0.187 for ^{100}Mo and ^{116}Cd, respectively.These values yield for the log ft values of the intermediate odd-odd nuclei results which deviate much from the corresponding experimental data. For this reason we fixed χ by fitting the log ft value characterizing the β^+/EC process of the intermediate nuclei. Results for χ obtained in this way are slightly different from those provided by the expression (7.2). The parameter χ_1 was fixed such that the log ft value characterizing the β^- decay of the intermediate odd-odd nuclei is close to the corresponding experimental data. The results obtained for the nuclei involved in the two processes are given in Table 1.

We note that the ISR is satisfied both for mother and daughter nuclei. In our calculation the ISR is sensitive to the dimension of the single particle basis. Indeed, chosing a basis of a smaller dimmension the ISR would be underestimated. Another parameters which influence the magnitude of ISR are the pairing strengths. Indeed ISR is increasing by increasing G_n or decreasing G_p. Variation of χ and χ_1 does not affect much ISR. However by varying these parameters the single beta strengths are modified according to the interaction nature. The strength is transfered to the lower energy by the attractive interaction (χ_1) and pushed up by the repulsive one (χ).

Using these input data we calculated the distribution of the $\beta^{\pm}$ strengths with the result shown in Fig.1. The energy intervals where both distributions are large, contribute significantly to the double beta transition amplitude. In plotting the β^+ strength we ignored the values smaller than 0.01. The $\beta^{\mp}$ strengths are fragmented among the $pnQRPA$ states, reflecting the fact that the single particle states are deformed. Note that the first peak for the β^- strength is the highest one while the one centered at higher energy has a large width and a fine substructure. The low energy peak is mainly determined by the attractive two-body interaction while the broad peak, i.e. the GT giant resonance, by the GT interaction. The β^+ strength is small in magnitude and less fragmented than the β^- strength. Also we note that the highest energy peak is the largest one.

Calculating first the GT transition amplitude and then the Fermi integral with $G_A =$

	M_{GT}	$T_{1/2}$[yr]				
	[MeV^{-1}]	present	Exp.	Ref.[18]	Refs. [30,31]	Refs.[32,33,34]
^{100}Mo$\rightarrow^{100}$Ru	0.221	$8.79\cdot10^{18}$	$(8.0\pm0.16)\cdot10^{18}$ a)	$4.4\cdot10^{18}$	$2.9\cdot10^{18}$ 3)	$1.8\cdot10^{18}$ 1)
			$(0.115^{+0.03}_{-0.02})\cdot10^{20}$ b)			
			$0.033^{+0.02}_{-0.01}\cdot10^{20}$ c,d)			
^{116}Cd$\rightarrow^{116}$Sn	0.15998	$2.02\cdot10^{19}$	$(3.2\pm0.3)\cdot10^{19}$ e)	$3.86\cdot10^{19}$	$5.1\cdot10^{19}$ 5)	$8.3\cdot10^{18}$
						$3.75\cdot10^{19}$ 6)

TABLE II: The Gamow-Teller amplitude for the $2\nu\beta\beta$ decay, in units of MeV^{-1}, and the corresponding half life ($T_{1/2}$), in units of yr, are listed for two ground to ground transitions. The experimental half lives for the transitions of ^{100}Mo (a) Ref.[26], b) Ref.[27] c) Ref.[28], d) Ref.[29])), ^{116}Cd (e) Ref.[36]), are also given. In the second last column the results reported in Refs. [30] 3) and [31] 5) are given. Comparison is also made with the theoretical results from the last column reported in Refs.[32] ($^{1)}$),[33] (unmarked) and [34] ($^{6)}$).

1.254, as in Ref.[4], we obtained the results given in Table II.

From Table II, one may see that the present approach provides for the half life of the double beta decay, values which are quite close to the experimental data. The results are compared with other theoretical calculations using different formalisms. Thus, in Ref.[18] a schematic Gamow-Teller proton-neutron interaction, in the particle-hole and particle-particle channels, is treated within a projected spherical single particle basis by a $pnQRPA$ approach. The results correspond to a large value of the parameter g_{pp}. Indeed, for the two nuclei considered here, ^{100}Mo and ^{116}Cd, the parameters (χ, g_{pp}) are (0.06,1.6) and (0.238,1.68) respectively. In Refs.[30, 31], a realistic Bonn-interaction is treated by a higher $pnQRPA$ approach, using a single particle basis corresponding to the Woods-Saxon potential. Of course, the ISR is not obeyed given the fact that a higher pnQRPA approach is used. The results shown in the last column were obtained using for proton-neutron interaction the Paris potential in both the ph and pp channel and a $pnQRPA$ formalism. It is worth mentioning that although within the $pnQRPA$ approach the ISR is obeyed the large value for the pp interaction strength raises the question whether the used formalism is still valid. On the other hand the higher $pnQRPA$ approach yielding the results shown in Table II on the column of Refs.[30, 31], leads necessarily to a violation of the ISR. We remark

Nucleus	^{100}Mo	^{116}Cd
$0.6\sum B(GT)_-$	28.96	36.2
$\sum B(GT)_-^{Exp}$	26.69	32.7

TABLE III: The experimental summed strength for the β^- transition (second row) is compared with the corresponding theoretical values quenched by a factor 0.6, for the two considered double beta emitters.

that these week points are not present in the present formalism. In Ref.[37] the standard renormalized $pnQRPA$ was applied for calculating the rate of double beta transitions of many isotopes. Unfortunately, the list does not include the isotopes considered here and consequently we cannot compare the predictions presented here with those from the quoted reference. Another fully renormalized procedure which obeys the ISR was formulated in Ref.[38]. The dependence of single beta decay strengths on the particle-particle interaction was studied numerically in Ref.[39]. By contrast, in our case the pp interaction does not contribute to the ground state correlations. Actually for this reason we replaced it by a dipole pairing interaction. This interaction brings important contributions to the backward going RPA amplitudes. However, even if this two-body interaction is missing the ground state correlations would not vanish due to the presence of the amplitudes $\mathcal{F}$ in the phonon operator expression. At its turn this is caused by the split of the pn dipole configurations in two orthogonal subspaces, one associated to the β^- transition while the other one to the β^+ decay. Also the ISR is obeyed. We added however the dipole pairing interaction in order to describe the log ft values associated to the single beta transitions of the intermediate odd-odd nuclei. The eigenvalue equations and the restrictions for the averaged quasiparticle number operators, in the two renormalization approaches, are different from each other.

Another experimental result concerns the summed strength for the β^- transition, denoted, conventionally, by $\sum B_{GT-}$. The experimental value of this sum covers only a fraction of the sum rule limit of $3(N-Z)$. Therefore in order to get a fair comparison of the calculated and measured quantity, we have to quench the calculated strength by a factor 0.6 in order to account for the missing experimental strength. The results are presented in Table III.

The intermediate odd-odd nucleus, ^{100}Tc, can perform the transition β^+/EC, feeding ^{100}Mo, or the β^- transition to ^{100}Ru. The same is true for ^{116}In which by means of a

Excited states	^{100}Tc				^{116}In			
	Ex[MeV]		B(GT)		Ex[MeV]		B(GT)	
	Exp.	Th.	Exp.	Th.	Exp.	Th	Exp.	Th.
GTR1	13.3	11.16	23.1± 3.8	15.63	14.5	12.37	25.8± 4.1	18.9
GTR2	8.0	8.05	2.9±0.5	5.87	8.9	7.87	6.6±1.1	7.2

TABLE IV: The strengths B(GT) of the single β^- transitions from the mother nuclei to the intermediate odd-odd nuclei excited in the states of the two components, GTR1 and GTR2, of the GT giant resonance are listed. The experimental (Exp.) and theoretical (Th.) values for the centroid energies of the two resonances are also specified.

EC process may undergo to ^{116}Cd while through a β^- decay can feed the nucleus ^{116}Sn. The measured $\log ft$ values for these transitions, are given in Table 1. The corresponding theoretical results are obtained by means of the expression:

$$ft_{\mp} = \frac{6160}{[{}_l\langle 1_1||\beta^{\pm}||0\rangle_l g_A]^2},\ l = i, f. \tag{7.3}$$

In order to take account of the effect of distant states responsible for the "missing strength" in the giant GT resonance [4] we chose $g_A = 1.0$. As we already mentioned, these single beta transitions were used as to fix the strengths of the two body dipole-dipole interactions.

The matrix elements involved in the double beta transition amplitude of the two emitters, ^{100}Mo and ^{116}Cd, have been experimentally investigated in Ref.[36]. The first matrix elements, describing the transitions $0_i^+ \rightarrow 1^+$, were obtained from the reactions ^{100}Mo(^{3}He,t)^{100}Tc and ^{116}Cd(^{3}He,t)^{116}In respectively, at $\theta_t \approx 0^0$, while the matrix elements for the $1^+ \rightarrow 0_f^+$ were derived from the known log ft value. For both cases the strength of the first β^- transition exhibits two bumps, one broad and called GTR1 while the second one less spread, located at lower energy and called GTR2. The centroid energies of the two resonances as well as the strength carried by each of them are compared with the theoretical results obtained with our approach, in Table IV.

From the mentioned Table we notice that the theoretical centroid energy for GTR1 is by about 2 MeV smaller than the experimental data. Moreover, the calculated strength carried by GTR1 is smaller than the corresponding experimental data. A reason for such a discrepancy might be the relative values for the attractive and repulsive dipole-dipole interaction intensities which favors the transfer of strength from the GTR states to the states

from the resonance of GTR2 and those from around 5 MeV. It is an open question whether these deviations could be washed out by a better fitting procedure or they constitute the price we have to pay for restoring the gauge symmetry. Note that the centroid for GTR2 of the ^{100}Mo$\rightarrow$ ^{100}Tc transition is fairly well described by our approach. As for the transition ^{116}Cd $\rightarrow^{116}$In the predicted centroid of GTR2 is 1 MeV lower in energy than the corresponding experimental data. The calculated strengths for GTR2 are larger than the corresponding experimental data. It is worth mentioning that the summed strength of the two resonances, GTR1 and GTR2, is reasonable close to the corresponding experimental data. A specific feature for our formalism consists of that states around the GTR1 centroid contribute to the peak seen in the strength distribution for the single β^+ decay of the daughter nuclei. Also for both nuclei it seams that the resonance around 5MeV for the β^- decay does not contribute at all to the double beta decay. Indeed, in this region the matrix element for the second leg of transition is almost vanishing. In this respect we notice that in the case of ^{116}Cd, a bunch of states below 2 MeV contribute to the double beta decay rate.

VIII. CONCLUSIONS

Summarizing the results of this paper, one may say that restoring the gauge symmetry from the fully renormalized pnQRPA, one obtains a realistic description of the transition rate and moreover the ISR is obeyed. As shown in this paper, it seems that there is no need to include the *pp* interaction in the many body treatment of the process. Indeed, in the framework of a FRpnQRPA approach this interaction violates the total number of particle and consequently the gauge projection process makes it ineffective.

Note that the hypothesis saying that the double beta process consists of two successive single β^- decays requires a consistent description of the double beta and single beta processes. In our formalism, actually, this feature is met since the rate of the double beta decay and the log ft values associated to the single beta decays of the intermediate odd-odd nuclei are realistically described. Another issue which is worth to be mentioned refers to the chain of approximations of the many-body Hamiltonian. A measure of the consistency of all these approximations is the ISR which in our case is satisfied to a high accuracy.

The attractive interaction of *ph* dipole-pairing type is responsible for the ground state correlations. To a less extent these are caused by the $\mathcal{F}$ components of the new phonon

operator. The projection of gauge is essential for restoring the ISR. The gauge projection of the pnQRPA was previously achieved in Ref.[35] where the ISR is anyway satisfied within the unprojected picture. By contrast therein the effect of projection is small.

The $GPFRpnQRPA$ equations consist of $FRpnQRPA$ equation (5.12) supplemented by Eqs. (5.6) and (5.17) which must be simultaneously solved by an iteration procedure. Since the two-body interaction is a separable interaction, Eq.(5.2) may be replaced by the dispersion equation (A.1) for energies and Eqs.(A.3) for the four phonon amplitudes. An extensive study of all existent data, with the formalism described in this paper, will be presented in a subsequent work.

Acknowledgment. This work was supported by the Romanian Ministry for Education Research Youth and Sport through the CNCSIS projects ID-33/2007 and ID-1038/2009.

IX. APPENDIX A

The compatibility condition for the homogeneous system of equations determining the phonon amplitudes yields two dispersion equations for ω:

$$\begin{aligned} & 4(\chi^2-\chi_1^2)\left[\mathcal{R}_-^+-\mathcal{R}_+^-\right]\left[\mathcal{R}_+^+-\mathcal{R}_-^-\right] \\ & +4\chi\left[\sum_{\mathcal{S}_+}\frac{T_{pn}^2E^{ren}(pn)}{\omega^2-(E^{ren}(pn))^2}+\sum_{\mathcal{S}_-}\frac{T_{pn}^2|E^{ren}(pn)|}{\omega^2-|E^{ren}(pn)|^2}\right]=1, \end{aligned} \tag{A.1}$$

with

$$\mathcal{R}_\pm^+=\sum_{(p_1n_1)\in\mathcal{S}_+}\frac{T_{p_1n_1}^2}{\omega\pm E^{ren}(p_1n_1)},\ \ \mathcal{R}_\pm^-=\sum_{(p_1n_1)\in\mathcal{S}_-}\frac{T_{p_1n_1}^2}{\omega\pm|E^{ren}(p_1n_1)|}. \tag{A.2}$$

The phonon amplitudes can be analytically determined. Indeed, the $GPFRpnQRPA$ equations yield the following expressions for the four amplitudes;

$$\begin{aligned} X(pn) &= 2\frac{T_{pn}^2}{\omega-E^{ren}(pn)}(\chi\mathcal{X}-\chi_1\mathcal{Y}),\ W(pn)=-2\frac{T_{pn}^2}{\omega+|E^{ren}(pn)|}(\chi\mathcal{X}-\chi_1\mathcal{Y}), \\ Z(pn) &= 2\chi\frac{T_{pn}^2}{\omega-|E^{ren}(pn)|}(\chi\mathcal{Y}-\chi_1\mathcal{X}),\ Y(pn)=-2\chi\frac{T_{pn}^2}{\omega+E^{ren}(pn)}(\chi\mathcal{Y}-\chi_1\mathcal{X}) \end{aligned} \tag{A.3}$$

The constant factors $\mathcal{X}$ and $\mathcal{Y}$ have the expressions:

$$\begin{aligned} \mathcal{X} &= \sum_{\mathcal{S}_+}T_{p_1n_1}X(p_1n_1)+\sum_{\mathcal{S}_-}T_{p_1n_1}W(p_1n_1), \\ \mathcal{Y} &= \sum_{\mathcal{S}_+}T_{p_1n_1}Y(p_1n_1)+\sum_{\mathcal{S}_-}T_{p_1n_1}Z(p_1n_1). \end{aligned} \tag{A.4}$$

The two factors are related by:

$$\mathcal{Y} = \frac{1}{\chi_1}\left[\chi - \frac{1}{2(\mathcal{R}_-^+ - \mathcal{R}_+^-)}\right]\mathcal{X} \equiv \mathcal{U}\mathcal{X}. \tag{A.5}$$

Finally the independent constant factor $\mathcal{X}$ is determined from the normalization condition of the phonon operator. The result is:

$$\begin{aligned}\mathcal{X}^{-2} \;=\; & 4\left[\sum_{\mathcal{S}_+}\frac{T_{pn}^2}{(\omega - E^{ren}(pn))^2} - \sum_{\mathcal{S}_-}\frac{T_{pn}^2}{(\omega + |E^{ren}(pn)|)^2}\right](\chi - \chi_1\mathcal{U})^2 \\ & +\; 4\left[\sum_{\mathcal{S}_-}\frac{T_{pn}^2}{(\omega - |E^{ren}(pn)|)^2} - \sum_{\mathcal{S}_+}\frac{T_{pn}^2}{(\omega + E^{ren}(pn))^2}\right](\chi_1 - \chi\mathcal{U})^2 .\end{aligned} \tag{A.6}$$

Having $\mathcal{X}$ and $\mathcal{Y}$ determined the phonon amplitudes are readily obtained by means of equations A.3.

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