Toward a new description of the intrinsic variables and the shape phase transition

A. A. Raduta[a),b)], A. C. Gheorghe[(b)], P. Buganu[b)] and Amand Faessler[c)]

[b)] *Department of Theoretical Physics and Mathematics,*

Bucharest University, POBox MG11, Romania

[a)] *Institute of Physics and Nuclear Engineering,*

Bucharest, POBox MG6, Romania and

[c)] *Institut fuer Theoretische Physik, der Universitaet*

Tuebingen, auf der Morgenstelle 14, Germany

Abstract

A new description of β and γ dynamic variables is proposed. The deformation γ is described by spheroidal periodic functions, while β by a set of functions satisfying a Schrödinger equation involving the Davidson's potential. In this way some drawbacks of the $X(5)$ model are removed. The proposed model goes to $X(5)$ in the limit of $|\gamma|$-small. Results of numerical applications to ^{150}Nd, ^{154}Gd and ^{192}Os are in good agreement to the experimental data. Comparison with $X(5)$ calculations suggests that the present approach provides a quantitative better description of the data.

PACS numbers: : 21.10.Re, 23.20.Lv, 21.60. Ev

I. INTRODUCTION

Since the liquid drop model was developed [1], the quadrupole shape coordinates were widely used by both phenomenological and microscopic formalisms to describe the basic properties of nuclear systems. Based on these coordinates, one defines quadrupole boson operators in terms of which model Hamiltonians and transition operators are defined. Since the original spherical harmonic liquid drop model was able to describe only a small amount of data for spherical nuclei, several improvements have been added. Thus, the Bohr-Mottelson model was generalized by Faessler and Greiner[2] in order to describe the small oscillations around a deformed shape which results in obtaining a flexible model, called vibration rotation model, suitable for the description of deformed nuclei. Later on [3] this picture was extended by including anharmonicities as low order invariant polynomials in the quadrupole coordinates. With a suitable choice of the parameters involved in the model Hamiltonian the equipotential energy surface may exhibit several types of minima [4] like spherical, deformed prolate, deformed oblate, deformed triaxial, etc. To each equilibrium shape, specific properties for excitation energies and electromagnetic transition probabilities show up. Due to this reason, one customarily says that static values of intrinsic coordinates determine a phase for the nuclear system. A weak point of the boson description with a complex anharmonic Hamiltonian consists of the large number of the structure parameters which are to be fitted. A much smaller number of parameters is used by the coherent state model (CSM) [5] which uses a restricted collective space generated through angular momentum projection by three deformed orthogonal functions of coherent type. The model is able to describe in a realistic fashion transitional and well deformed nuclei of various shapes including states of high and very high angular momentum. Various extensions to include other degrees of freedom like isospin [6], single particle[7] or octupole [8] degrees of freedom have been formulated[9].

It has been noticed that a given nuclear phase may be associated to a certain symmetry. Hence, its properties may be described with the help of the irreducible representation of the respective symmetry group. Thus, the gamma unstable nuclei can be described by the $O(6)$ symmetry [10], the gamma triaxial nuclei by the rigid triaxial rotor $D2$ symmetry [11], the symmetric rotor by the $SU(3)$ symmetry and the spherical vibrator by the $U(5)$ symmetry. Thus, even in the 50's, the symmetry properties have been greatly appreciated. However, a big push forward was brought by the interacting boson approximation (IBA)

[12, 13], which succeeded to describe the basic properties of a large number of nuclei in terms of the symmetries associated to a system of quadrupole (d) and monopole (s) bosons which generate a $U(6)$ algebra. The three limiting symmetries $U(5)$, $O(6)$, $SU(3)$ mentioned above, are dynamic symmetries for $U(6)$. Moreover, for each of these symmetries a specific group reduction chain provides the quantum numbers characterizing the states, which are suitable for a certain region of nuclei. Besides the virtue of unifying the group theoretical descriptions of nuclei exhibiting different symmetries, the procedure defines very simple reference pictures for the limiting cases. For nuclei lying close to the region characterized by a certain symmetry, the perturbative corrections are to be included.

In Ref. [14],a new classification scheme was provided, all nuclei being distributed on the border of a symmetry triangle. The vertices of this triangle symbolize the $U(5)$ (vibrator), $O(6)$ (gamma soft) and $SU(3)$ (symmetric rotor), while the legs of the triangle denote the transitional region. Properties of nuclei lying far from vertices are difficult to be explained since the states have some characteristics of one vertex while some others are easy to be described by using the adjacent symmetry. When the mixture of the extreme symmetries is maximal, one says that the critical point for the transition from one phase to another has been reached. In Ref. [15, 16], it has been proved that on the $U(5) - O(6)$ transition leg there exists a critical point for a second order phase transition while the $U(5) - SU(3)$ leg has a first order phase transition.

Recently, Iachello [17, 18] pointed out that these critical points correspond to distinct symmetries, namely $E(5)$ and $X(5)$, respectively. For the critical value of an ordering parameter, energies are given by the zeros of a Bessel function of half integer and irrational indices, respectively.

This procedure has been often used in the past, in various contexts. For example, when the classical electromagnetic field is quantized it is confined inside a sphere of radius R on whose surface the solution of the field equation vanishes. Moreover, the magnitude of the radius determine the photon energy. Another interesting fact to be recalled refers to the property of the RPA (random phase approximation) solution of a many body Hamiltonian involving a spherical shell model mean field and a attractive separable Q.Q interaction. If we increase continuously the strength of the long range interaction one reaches a critical value when the first solution, ω, of the RPA equation vanishes. This is a critical point for a phase transition from spherical to deformed equilibrium shape for the mean field. Since for the critical value of the interaction strength $\omega = 0$ it results that the phonon operator becomes a generator of a symmetry (due to the vanishing commutator with the many body Hamiltonian). Therefore the transition from a spherical to a deformed shape symmetry exhibits a critical point which correspond to another symmetry.

The description of low lying states in terms of Bessel functions was used first by Jean and Willet [10], but the interesting feature saying that this is a critical picture in a phase transition and defines a new symmetry, was indeed advanced first in Ref.[17].

Representatives for the two symmetries have been experimentally identified. To give an example, the relevant data for ^{134}Ba [19] and ^{152}Sm [20] suggest that they are close to the $E(5)$ and $X(5)$ symmetries, respectively. Another candidate for E(5) symmetry, proposed by Zamfir *et al.* [22] is ^{102}Pd. Using a simple IBA Hamiltonian, in Ref.[23], the low lying spectrum of ^{108}Pd is realistically described. Comparing the E(5) predictions with the experimental data concerning energy ratios in the ground band and the normalized E2 transition probabilities for the states 4^+ and 0_2^+, one concludes that this nucleus is a good E(5) candidate. However, in order to decide which Pd isotope is closer to an E(5) behavior, further investigations are necessary. A systematic search for $E(5)$ behavior in nuclei has been reported in Ref.[21].

Short after the pioneering papers concerning critical point symmetries appeared, some other attempts have been performed, using other potentials like Coulomb, Kratzer [24] and Davidson potentials [25]. These potentials yield also Schrödinger solvable equations and the corresponding results may be interpreted in terms of symmetry groups.

In Ref.[27] we advanced the hypothesis that the critical point in a phase transition is state dependent. We tested this with a hybrid model for ^{134}Ba and ^{104}Ru.

The departure from the gamma unstable picture has been treated by several authors whose contributions are reviewed by Fortunato in Ref.[28]. The difficulty in treating the gamma degree of freedom consists in the fact that this variable is coupled to the rotation variables. A full solution for the Bohr-Mottelson Hamiltonian including an explicit treatment of gamma deformation variable can be found in Refs.[29, 30]. Therein, we treated separately also the gamma unstable and the rotor Hamiltonian. A more complete study of the rotor Hamiltonian and the distinct phases associated to a tilted moving rotor is given in Ref. [31].

The treatment of the γ variable becomes even more complicated when we add to the liquid drop Hamiltonian a potential depending on β and γ at a time. To simplify the starting problem related to the inclusion of the gamma variable one uses model potentials which are sums of a beta and a gamma depending potentials. In this way the nice feature for the beta variable to be decoupled from the remaining 4 variables, specific to the harmonic liquid drop, is preserved. Further the potential in gamma is expanded either around to $\gamma = 0$ or around $\gamma = \frac{\pi}{2}$. In the first case if only the singular term is retained one obtains the infinite square well model described by Bessel functions in gamma. If the γ^2 term is added to this term, the Laguerre functions are the eigenstates of the approximated gamma depending Hamiltonian, which results in defining the representation of the X(5) symmetry group.

Note that any approximation applied to the γ-Hamiltonian modifies automatically the differential equation for β. Indeed, the centrifugal term $\tau(\tau+3)/\beta^2$ disappears but another one is expected to come from the $\beta-\gamma$ coupling after some approximations are performed.

The drawback of these approximation consists in that the resulting function are not periodic as the starting Hamiltonian is. Moreover, they are orthonormalized on unbound intervals although the underlying equation was derived under the condition of $|\gamma|$ small. Moreover, the scalar product for the space of the resulting functions is not defined based on the measure $|\sin 3\gamma|d\gamma$ as happens in the liquid drop model. Under these circumstances it happens that the approximated Hamiltonian in γ looses its hermiticity.

In a previous short publication [32] we proposed a scheme where the gamma variable is described by a solvable Hamiltonian whose eigenstates are spheroidal functions which are periodic. Here we give details about the calculations and describe some new numerical applications. Moreover, we complete our formalism by treating the β variable by a Schrödinger equation associated to the Davidson's potential.

By the numerical applications we want to see whether curing the mentioned drawbacks for gamma and beta variables respectively, would bring substantial quantitative corrections to energies and E2 transitions given by the $X(5)$ formalism. Even if the found corrections are not dramatical the virtue of the proposed formalism to extend the $X(5)$ description to a theory which is consistent with the symmetries of the starting Hamiltonian as well with some basic principles of Quantum Mechanics concerning the hermiticity property of the model Hamiltonian remains an important achievement of the present paper.

The above objectives are reached according to the following plan. The starting Hamiltonian is presented in Section II, where the separability conditions are discussed. Several methods for treating β are described in Subsection A while the variable γ is studied in Subsection B. The specific procedure of the present paper is presented in Section III. Numerical applications are given in Section IV. The final conclusions are summarized in Section V.

II. THE STARTING HAMILTONIAN

Written in the intrinsic frame of reference, the original Bohr-Mottelson Hamiltonian has the expression:

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}\right] + V(\beta,\gamma), \tag{2.1}$$

where the dynamic deformation variables are denoted by β and γ while the intrinsic angular momentum components by Q_k, with $k = 1, 2, 3$. Within the liquid drop model the potential energy depends quadratically on β. Here we assume that the potential energy depends on both deformation variables, beta and gamma. If the potential energy term is depending on deformation variables in a separable manner:

$$V(\beta,\gamma) = V(\beta) + U(\gamma), \tag{2.2}$$

the eigenvalue equation associated to H (2.1) can be separated in two parts, one equation describing the beta variable and the other one the gamma deformation and the Euler angles $\Omega = (\theta_1, \theta_2, \theta_3)$.

Separation of variables is based on two approximations [48, 49]: i) restriction to small values of γ, i. e. $|\gamma| \ll 1$; ii) replacing the factor $1/\beta^2$ by $1/\langle\beta^2\rangle$ in the terms involved in the equation for γ. The diagonalization of the Bohr-Mottelson Hamiltonian shows that the first approximation is valid for large γ stiffness while the second one for small γ stiffness [49].

A complete separation of equations for the two variables, β and γ is possible if we choose the potential

$$V(\beta,\gamma) = V(\beta) + U(\gamma)/\beta^2. \tag{2.3}$$

In this way the approximation consisting of replacing β^2 by $\langle\beta^2\rangle$ is avoided but the restriction to the case $|\gamma| \ll 1$ is still kept. Thus, the theory involves one parameter which is the γ stiffness which affects the excitation energies in both the beta and gamma bands.

In what follows the two equations will be considered separately.

A. The treatment of β variable

The solvable models for the variable β presented here have been used by E(5) formalisms which ignore the potential in γ. Considering the potential in γ, of course, the picture for β will change. However, as we shall see later on, it is very easy to derive analytically the energies and wave functions associated to β from the corresponding results of the E(5) descriptions. Actually this is the motivation for reviewing the beta solvable models here.

The equation in β is:

$$\left[-\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\Lambda}{\beta^2}+u(\beta)\right]f(\beta)=\epsilon f(\beta), \tag{2.4}$$

where Λ is the eigenvalue of the Casimir operator of the $SO(5)$ group. This is related with the seniority quantum number τ, by $\Lambda=\tau(\tau+3)$. The 'reduced' potential $u(\beta)$ and energy ϵ are defined as:

$$E=\frac{\hbar^2}{2B}\epsilon,\quad U=\frac{\hbar^2}{2B}u. \tag{2.5}$$

where E denotes the eigenvalue of the Hamiltonian H corresponding to the potential $U(\beta)$. Here we mention the most used potentials for β

1. *The case of $u(\beta)=\beta^2$*

A full description of the eigenstates of the Bohr-Mottelson Hamiltonian satisfying the symmetry $U(5)\supset SO(5)\supset SO(3)\supset SO(2)$, may be found in Refs.[29]. In particular, the solution of the radial equation (2.4) with $u(\beta)=\beta^2$ is easily obtained by bringing first Eq.(2.4) to the standard Schrödinger form by changing the function f to ψ by:

$$\psi(\beta)=\beta^2 f(\beta). \tag{2.6}$$

The equation obeyed by the new function ψ, is:

$$\frac{d^2\psi}{d\beta^2}+\left[\epsilon-\beta^2-\frac{(\tau+1)(\tau+2)}{\beta^2}\right]\psi=0. \tag{2.7}$$

This equation is analytically solvable. The solution is:

$$\psi_{n\tau}(\beta) = \sqrt{\frac{2(n!)}{\Gamma(n+\tau+5/2)}}L_n^{\tau+3/2}(\beta^2)\beta^{\tau+2}\exp(-\beta^2/2), \tag{2.8}$$

$$\epsilon_n = 2n+\tau+5/2,\quad n=0,\ 1,\ 2,\ ...;\tau=0,1,2,3,... \tag{2.9}$$

where L_n^ν denotes the generalized Laguerre polynomials. The number of polynomial nodes is denoted by n and is related to the number of the quadrupole bosons (N) in the state, by: $N = 2n + \tau$. Consequently, the initial equation (2.4) has the solution

$$f_{n\tau} = \beta^{-2}\psi_{n\tau} \ . \tag{2.10}$$

The spectrum, given by Eq. (2.9), may be also obtained by using the unitary representation of the $SU(1,1)$ group with the Bargman index $k = (\tau + 5/2)/2$. Indeed, the standard generator for $SU(1,1)$ are:

$$K_0 = \frac{1}{4}H_0, \quad K_\pm = \frac{1}{4}\left[\frac{(\tau+1)(\tau+2)}{\beta^2} - (\beta \pm \frac{d}{d\beta})^2\right], \tag{2.11}$$

where

$$\begin{aligned} H_0 &= -\frac{d^2}{d\beta^2} + \beta^2 + \frac{(\tau+1)(\tau+2)}{\beta^2}, \\ H_0\psi_n &= \epsilon_n\psi_n. \end{aligned} \tag{2.12}$$

The commutation relations of H_0 are:

$$[K_-, K_+] = -\frac{1}{2}H_0, \ \ [K_\pm, H_0] = \pm 4K_\pm. \tag{2.13}$$

2. Davidson's potential

Another potential energy term in β which yields also a solvable model is due to Davidson [33]:

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}. \tag{2.14}$$

This potential has been used by several authors in different contexts [25, 34, 35]. The potential has been used by Bonatsos *at al*[49], to describe the dynamic deformation variable β. For this potential the above equations (2.6-10) hold for τ replaced [25] by

$$\tau' = -\frac{3}{2} + \left[\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4\right]^{1/2}. \tag{2.15}$$

In particular, the excitation energies have the expressions:

$$E_{n,\tau} = 2n + 1 + \left[\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4\right]^{1/2}. \tag{2.16}$$

The factor β_0^4 is considered to be a free parameter which is to be determined variationally for each angular momentum, as was suggested in Ref. [25]:

$$\frac{d^2 R_L^{(g)}}{d\beta_0^2} = 0, \tag{2.17}$$

where R denotes the ratio of the excitation energy of the ground band state L^+ and the excitation energy of the state 2_g^+.

3. *Five dimensional infinite well*

Now, let us turn our attention to the situation considered by Iachello in Ref.[17], where the potential term associated to the spherical to gamma unstable shape transition is so flat that it can be mocked up as a infinity square well

$$u(\beta) = \begin{cases} 0, & \beta \leq \beta_w, \\ \infty, & \beta > \beta_w. \end{cases} \tag{2.18}$$

A more convenient form for the equation in β, is obtained through the function transformation:

$$\varphi(\beta) = \beta^{3/2} f(\beta), \tag{2.19}$$

The equation for φ is

$$\frac{d^2\varphi}{d\beta^2} + \frac{1}{\beta}\frac{d\varphi}{d\beta} + \left[\epsilon - u(\beta) - \frac{(\tau + 3/2)^2}{\beta^2}\right]\varphi = 0. \tag{2.20}$$

Changing the variable β to z by

$$z = k\beta, \quad k = \sqrt{\epsilon} \tag{2.21}$$

and denoting with $\widetilde{\varphi}(z) = \varphi(\beta)$ the function of the new variable, one arrives at:

$$\frac{d^2\widetilde{\varphi}}{dz^2} + \frac{1}{z}\frac{d\widetilde{\varphi}}{dz} + \left[1 - \frac{(\tau + 3/2)^2}{z^2}\right]\widetilde{\varphi} = 0. \tag{2.22}$$

This equation is analytically solvable, the solutions being the Bessel functions of half integer order, $J_{\tau+3/2}(z)$. Since for $\beta > \beta_w$ the function $\widetilde{\varphi}$ is equal to zero, the continuity condition requires that the solution inside the well must vanish for the value of β equal to β_w. This, in fact, yields a quantized form for the eigenvalue E. Indeed, let $x_{\xi,\tau}$ be the zeros of the Bessel function J_ν :

$$J_{\tau+3/2}(x_{\xi,\tau}) = 0, \quad \xi = 1, 2, ...; \tau = 0, 1, 2, ... \tag{2.23}$$

Then, due to the substitution introduced in Eqs.(2.12) and (2.4) one obtains:

$$E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \quad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_w}. \tag{2.24}$$

Concluding, the differential equation for the beta deformation corresponding to an infinite well potential provides the energy spectrum given by Eq.(2.24) and the wave functions:

$$f_{\xi,\tau} = C_{\xi,\tau} \beta^{-3/2} J_{\tau+3/2}(\frac{x_{\xi,\tau}}{\beta_w}\beta), \tag{2.25}$$

where $C_{\xi,\tau}$ is a normalization factor. Actually, these functions represent the irreducible representation of the group $E(5)$.

It is worth noticing that the spectra corresponding to E(5) and Davidson potentials become directly comparable by establishing the formal correspondence $n = \xi - 1$.

4. A hybrid model

In ref.[27] we advanced the idea that the critical point for a phase transition is depending on the nuclear state. Therefore the system may reach the critical point in a state of angular momentum J, but in a more excited state like $(J+1)^+$ the system could behave according to the initial nuclear phase.

According to Ref.[27] the potential energy in the beta variable is depending on angular momentum in the following way:

$$u(\beta) = \begin{cases} \beta^2, & \text{if} \quad 0 \le \beta < \infty, \quad L \le 2, \\ 0, & \text{if} \quad 0 \le \beta \le \beta_w, \quad L \ge 4, \\ \infty, & \text{if} \quad \beta_w < \beta < \infty, \quad L \ge 4. \end{cases} \tag{2.26}$$

The states of interest and their energies have the following expressions:

$$\begin{aligned} |L^+_{n\tau} M\rangle &= \sqrt{\frac{2n!}{\Gamma(n+\tau+5/2)}} \beta^\tau L_n^{\tau+3/2}(\beta^2) e^{-\beta^2/2} G^{LM}_{n\tau}(\gamma,\Omega), \\ E_{n\tau} &= \frac{\hbar^2}{2B}(2n+\tau+5/2), \quad (n,\tau) = (0,0),\ (0,1), \quad L = 2\tau, \\ |L^+_{\xi,\tau} M\rangle &= C_{\xi,\tau}\beta^{-3/2} J_{\tau+3/2}(\beta x_{\xi,\tau}/\beta_w) G^{LM}_{\xi-1,\tau}(\gamma,\Omega), \\ E_{\xi,\tau} &= \frac{\hbar^2}{2B}\frac{x^2_{\xi,\tau}}{\beta_w^2}, \quad (\xi,\tau) = (1,2),\ (1,3),\ (2,0). \end{aligned} \tag{2.27}$$

The factor functions depending on the beta variable are solutions of Eq.(2.4) with the reduced potential given by Eq.(2.26). The equation for γ deformation and Eulerian angles (Ω) has the solution $G^{LM}_{n\tau}$.

Note that in all treatments mentioned above we did not consider any potential in γ Due to this fact the spectra and wave functions are labeled by the seniority quantum number τ. This feature does not hold when we switch on the γ-depending potential and moreover impose variable separability by approximating the terms depending on γ

B. The description of γ degree of freedom

Let us consider the Hamiltonian

$$H = -\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + U(\gamma) + W(\gamma, Q), \tag{2.28}$$

where U is a periodic function in γ with the period equal to 2π and

$$W(\gamma, Q) = \frac{1}{4}\sum_{k=1}^{3}\frac{1}{\sin^2\left(\gamma - \frac{2\pi}{3}k\right)}Q_k^2 \tag{2.29}$$

with Q_k denoting the components of the intrinsic angular momentum.

1. Violating some basic properties

Any approximation for the potential by expanding it in power series of γ alters the periodic behavior of the eigenfunction unless a special caution is payed. Moreover, the approximating Hamiltonian loses its hermiticity with respect to the scalar product defined with the measure for the gamma variable $|\sin(3\gamma)|d\gamma$.

We illustrate this by considering the case of a little more complex potential

$$U = u_1\cos(3\gamma) + u_2\cos^2(3\gamma). \tag{2.30}$$

Performing the change of function $\varphi = \sqrt{|\sin(3\gamma)|}\psi$, the eigenvalue equation $H\psi = E\psi$, becomes $\tilde{H}\varphi = 0$, with

$$\tilde{H} = \frac{\partial^2}{\partial\gamma^2} + \frac{9}{4}\left[1 + \frac{1}{\sin^2(3\gamma)}\right] - U - W + E. \tag{2.31}$$

We shall consider two situations:

A. Suppose that $|\gamma| \ll 1$. Expanding the terms in γ in power series up to the fourth order, one obtains:

$$
\begin{aligned}
U_4 &= u_1 + u_2 - 9\gamma^2\left(\frac{u_1}{2} + u_2\right) + 27\gamma^4\left(\frac{u_1}{8} + u_2\right), \\
W_4 &= \frac{1}{3}\left(1 + 2\gamma^2 + \frac{26\gamma^4}{9}\right)\left(Q_1^2 + Q_2^2\right) \\
&+ \frac{2\sqrt{3}\gamma}{9}\left(1 + 2\gamma^2\right)\left(Q_2^2 - Q_1^2\right) \\
&+ \frac{1}{4}\left(\frac{1}{\gamma^2} + \frac{1}{3} + \frac{\gamma^2}{15} + \frac{2\gamma^4}{189}\right)Q_3^2.
\end{aligned}
\tag{2.32}
$$

The low index accompanying U and W suggests that the expansions were truncated at the fourth order. Details about the approximations involved in the following derivation may be found in Appendix A.

Note that due to the term W, the equations of motion for the variable γ and and Euler angles are coupled together. Such a coupling term can in principle be handled as we did for the harmonic liquid drop in ref. [29, 30]. Here, we separate the equation for γ by averaging W_4 with an eigenfunction for the intrinsic angular momentum squared. The final result for H_4 is:

$$
\begin{aligned}
H_4 &= \frac{\partial^2}{\partial\gamma^2} + \frac{1}{4\gamma^2}\left(1 - \left\langle Q_3^2\right\rangle\right) + h_0 + h_2\gamma^2 + h_4\gamma^4 \\
&+ \frac{2\sqrt{3}\gamma}{9}\left(1 + 2\gamma^2\right)\left\langle Q_2^2 - Q_1^2\right\rangle, \\
h_0 &= E - \frac{1}{3}L(L+1) + \frac{1}{4}\left\langle Q_3^2\right\rangle - (u_0 + u_1 + u_2) + \frac{15}{2}, \\
h_2 &= -\frac{2}{3}L(L+1) - \frac{13}{20}\left\langle Q_3^2\right\rangle + \frac{9}{2}u_1 + 9u_2 + \frac{27}{20}, \\
h_4 &= -\frac{26}{27}L(L+1) - \frac{121}{126}\left\langle Q_3^2\right\rangle - \frac{27}{8}u_1 - 27u_2 + \frac{27}{14}.
\end{aligned}
\tag{2.33}
$$

where L denotes the angular momentum. If the average is made on the Wigner function D^L_{MK}, important simplifications are obtained since the following relations hold:

$$
\left\langle Q_2^2 - Q_1^2\right\rangle = 0, \quad \left\langle Q_3^2\right\rangle = K^2 \tag{2.34}
$$

Let us make the option for this situation. Note that H_4 contains a singular term in γ, at $\gamma = 0$, coming from the term coupling the intrinsic variable γ with the Euler angles. One could get rid of such a coupling term by starting with a potential in gamma containing a singular term which cancels the contribution produced by the W term. Thus, the new potential would be

$$U' = U + \frac{9K^2}{4\sin^2(3\gamma)}. \tag{2.35}$$

The corresponding fourth order expansion for the Hamiltonian is:

$$\begin{aligned} H'_4 &= \frac{\partial^2}{\partial\gamma^2} + \frac{1}{4\gamma^2} + h'_0 + h'_2\gamma^2 + h'_4\gamma^4, \\ h'_0 &= h_0 + K^2,\ h'_2 = h_2 + \frac{27}{20}K^2,\ h'_4 = h_4 + \frac{27}{14}K^2. \end{aligned} \tag{2.36}$$

Some remarks concerning the equation $H'_4\varphi = 0$ are worth to be mentioned:

i) If in this equation one ignores the γ^4 term, the resulting equation has the Laguerre functions as solutions and moreover the Hamiltonian exhibits the $X(5)$ symmetry.

ii) Note also that the Hamiltonian coefficients are different from those of Ref.[28]. The difference is caused by the fact that there, the expansion was not consistently achieved.

iii) Taking in the expanded potential $u_1 = u_2 = 0$ and ignoring, for γ small, the term $\frac{27}{20}K^2\gamma^2$, the resulting potential is that of an infinite square well which was treated by Iachello in Ref[18]. The solutions are, of course, the Bessel functions of half integer indices.

iv) Irrespective of the potential in γ, in the regime of $|\gamma|$ small a term proportional to γ^2 shows up due to the rotational Hamiltonian W. Therefore, even in the case the potential is taken as an infinite square well, of the form $1/\gamma^2$, the equation describing the γ variable admits a Laguerre function as solution and not, as might be expected, a Bessel function of semi-integer index. Amazingly, the potential in γ is also of Davidson type.

v) None of the mentioned solutions is periodic.

vi) Also the approximated Hamiltonians are not Hermitian in the Hilbert space of functions in gamma with the integration measure as introduced by the liquid drop model, i.e. $|\sin 3\gamma| d\gamma$.

B. The case $|\gamma - \pi/6| \ll 1$. Using the fourth order expansion in $y = |\gamma - \pi/6|$, given in Appendix A, one obtains a Hamiltonian similar to that given by Eq. (2.36):

$$H'_4 = \frac{\partial}{\partial\gamma^2} + h'_2\gamma^2 + h'_4\gamma^4 + 2\sqrt{3}y\left(1 + \frac{22\sqrt{3}}{3}y^2\right)\langle Q_3^2 - Q_2^2\rangle. \tag{2.37}$$

If $\langle Q_3^2 - Q_2^2 \rangle = 0$ and, moreover, one ignores the term in γ^4 the resulting equation in γ describes a harmonic oscillator. Again the eigenfunctions, i.e. the Hermite functions, are orthogonal on an unbound interval of γ, and not on $[0, 2\pi]$.

2. Toward an exact treatment which preserves periodicity and hermiticity

To overcome the principle problems mentioned above, we try first to avoid making approximations. Thus, let us consider the Hamiltonian given by Eq.(2.28) where instead of U we consider U' as defined by Eq.(2.35), and ignore for a moment W. Changing the variable $x = \cos 3\gamma$, the eigenvalue equation associated to this Hamiltonian becomes:

$$\left(1-x^2\right)\frac{\mathrm{d}^2 S}{\mathrm{d}\,x^2} - 2x\frac{\mathrm{d}S}{\mathrm{d}\,x} + \left(\frac{1}{9}(E - u_1 x - u_2 x^2) - \frac{K^2}{4(1-x^2)}\right) S = 0. \tag{2.38}$$

If $u_1 = u_2 = K = 0$, the solution of this equation is the Legendre polynomial P_n while $E = 9n(n+1)$. This case has been considered in Ref.[28]. For other particular choices of the coefficients u_1, u_2 defining the potential in gamma, the solution is readily obtained if one compares the above equation with that characterizing the spheroidal oblate functions[36]

$$\left(1-x^2\right)\frac{\mathrm{d}^2 S_{nm}}{\mathrm{d}\,x^2} - 2x\frac{\mathrm{d}S_{nm}}{\mathrm{d}\,x} + \left(\lambda_{nm} - c^2x^2 - \frac{m^2}{1-x^2}\right) S_{nm} = 0. \tag{2.39}$$

The prolate case is reached by changing $c \to$ ic.

For $c = 0$, the solutions of Eq.(2.39) are the associated Legendre functions P_n^m. For $c \neq 0$, S_{nm}, with m, n integers and $n \geq m \geq 0$, are linear series of these functions.

In the case $u_1 = 0$, the solution of Eq.(2.38) is identified as being the spheroidal function while the energy is simply related to λ_{mn}:

$$m = \frac{K}{2}, \quad c^2 = \frac{u_2}{9}, \quad \lambda_{nm} = \frac{1}{9}E. \tag{2.40}$$

For $|c|$ small the energies E_{nm} exhibits the asymptotic expansion

$$\begin{aligned} E_{nm} &\approx 9n(n+1) - \frac{2\left(n(n+1)+m^2-1\right)}{(2n-1)(2n+3)}u_2 \\ &+ \frac{1}{18}\frac{\left[(n-1)^2-m^2\right]\left(n^2-m^2\right)}{(2n-3)(2n-1)^3(2n+1)}u_2^2 \\ &- \frac{1}{18}\frac{\left[(n+1)^2-m^2\right]\left[(n+2)^2-m^2\right]}{(2n+1)(2n+3)^3(2n+5)}u_2^2. \end{aligned} \tag{2.41}$$

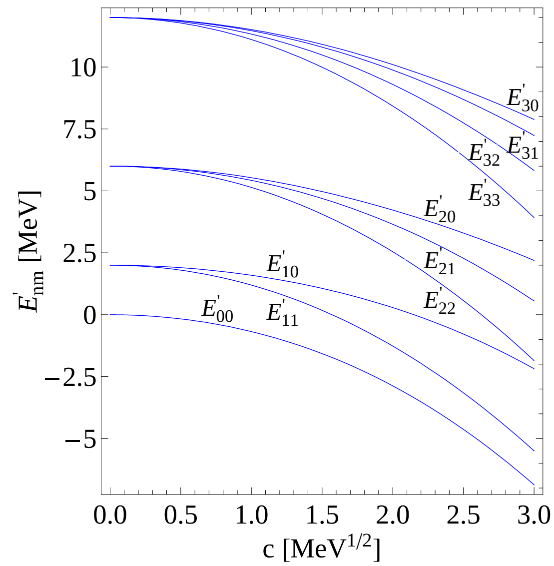


FIG. 1: (Color online).The spheroidal energy $E'_{nm} = \lambda_{nm} = E_{nm}/9$, for $0 \leq m \leq n \leq 3$ are plotted as functions of $c = \sqrt{u_2}/3$.

Eq.(2.41) considered for a fixed m but various n, defines a band. Similar expansions may be derived for $|c|$ large.

$$\begin{aligned} E_{nm} &\approx -u_2 + 3q\sqrt{u_2} + 9\left(m^2 - \frac{q^2+5}{8}\right) - \frac{27q}{64\sqrt{u_2}}(11 + q^2 - 32m^2), \\ q &= 2(n-m) + 1 \end{aligned} \tag{2.42}$$

We remark that the spectrum has a rotational behavior for small c, due to the term $n(n+1)$ while for large values of c it has an oscillator feature, the energy depending linearly on n.

If one needs the expansion up to the $1/c^2$ terms, the results for the first few energies are:

$$\begin{aligned} E_{11} &= 9\left(\frac{1}{4} - c^2 + c + \frac{5}{16c} + \frac{33}{64c^2}\right), \\ E_{21} &= 9\left(-\frac{3}{4} - c^2 + 3c + \frac{9}{16c} + \frac{135}{64c^2}\right), \\ E_{22} &= 9\left(\frac{13}{4} - c^2 + c + \frac{29}{16c} + \frac{177}{64c^2}\right), \\ E_{31} &= 9\left(-\frac{11}{4} - c^2 + 5c - \frac{5}{16c} + \frac{219}{64c^2}\right), \\ E_{32} &= 9\left(\frac{9}{4} - c^2 + 3c + \frac{81}{16c} + \frac{855}{64c^2}\right), \end{aligned}$$

$$E_{33} = 9\left(\frac{33}{4} - c^2 + c + \frac{69}{16c} + \frac{417}{64c^2}\right). \tag{2.43}$$

It is worth spending few words about Fig. 1 where the energies yielded by the spheroidal functions with the parameters specified by Eq.(2.40). Indeed for $c \to 0$ one notices some multiplet degeneracy which suggest a symmetry with respect to K, i.e. a rotation invariance of states of a given n. Increasing c the split in energy is similar to that in Nilsson [50] model when the energy is Ω dependent. The difference is that while in Nilsson model each deformed state is a superposition of states with different angular momentum, here the multiplet members are characterized by the same n. In this respect the feature shown in Fig.1 is similar to the one obtained with a spherical projected single particle basis [51]. In the region of large c, for a given large n the set of states of different m seem to form a band. On the other hand for a fixed m the set of states with different n is a band of equidistant energy levels.

C. Approximation which does not affect periodicity and hermiticity

Now, we shall focus on an approximate solution which preserves the periodicity in γ. For that purpose we consider the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + U(\gamma), \\ U(\gamma) &= u_1\cos 3\gamma + u_2\cos^2 3\gamma + \frac{K^2}{4\sin^2\gamma}. \end{aligned} \tag{2.44}$$

Changing the function by the transformation $\Psi = |\sin(3\gamma)|^{-1/2}\tilde{\Psi}$, for $\sin(3\gamma) \neq 0$, the eigenvalue equation for H becomes:

$$\left[\frac{\partial^2}{\partial\gamma^2} + E + \frac{9}{4} + \frac{9}{4\sin^2 3\gamma} - U(\gamma)\right]\tilde{\Psi} = 0. \tag{2.45}$$

Under the regime of $|\gamma|$ small, we take the $O(\gamma^3)$ expansion of the terms depending on γ and in the final expression approximate $\gamma \approx \sin\gamma$. In this way Eq.(2.45) becomes:

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\gamma^2} + a - 2q\cos 2\gamma - \frac{K^2-1}{4\sin^2\gamma}\right)\Phi = 0, \text{ with} \\ &q = \frac{1}{3} + \frac{9}{8}u_1 + \frac{9}{4}u_2, u = u_2 + \frac{347}{108}, a = E + \frac{10}{9}q + u. \end{aligned} \tag{2.46}$$

We suppose now that this equation is valid in the interval $[0, 2\pi]$. The equation (2.46) is just the trigonometric form of the spheroidal functions. The algebraic version is obtained by changing the variable $x = \cos\gamma$.

For $K = 1$ one obtains the Mathieu equation:

$$\left(\frac{\partial^2}{\partial\gamma^2} + a - 2q\cos 2\gamma\right)\Phi = 0. \tag{2.47}$$

There are two sets of solutions, one even and one odd denoted by $\Phi^+(a, q, \gamma)$ and $\Phi^-(a, q, \gamma)$, respectively. For $q = 0$, both solutions are periodic for any positive value of a.

$$\Phi^+(a, 0, \gamma) = \cos\left(\sqrt{a}\gamma\right),\ \Phi^-(a, 0, \gamma) = \sin\left(\sqrt{a}\gamma\right). \tag{2.48}$$

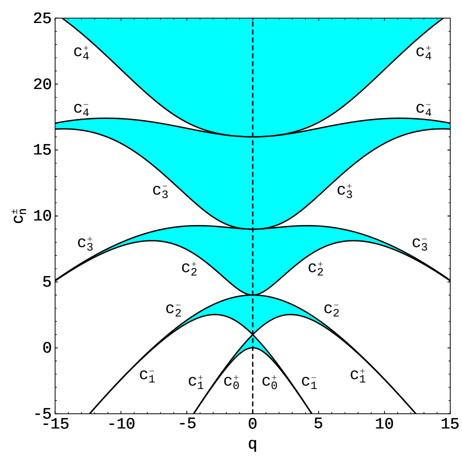


FIG. 2: (Color online)The characteristic curves $c_n^\pm$ are plotted as functions of q for several values of n.

For $q \neq 0$ the Mathieu functions are periodic in γ only for a certain set of values of a, called characteristic values. These are denoted by c_n^+ for even and c_n^- for odd functions, respectively. In the plane (a, q), the characteristics curves $c_n^\pm$ separate the stability regions, shown in Fig. 2 by gray color, from the non-stability ones, indicated by white color in the quoted figure. For $q = 0$ the equalities $c_n^\pm(0) = n^2$ hold. By means of Eq.(2.46) the characteristic values determine the energy E. Thus, the energy spectrum is given by $E_n^\pm - u$ with $E_n^\pm = c_n^\pm - \frac{10}{9}q$. The corresponding wave functions are the elliptic cosine and elliptic sine functions respectively:

$$\begin{aligned}
\Phi_0^+(q,\gamma) &= \frac{1}{\sqrt{2\pi}}ce_0(q,\gamma), \Phi_n^+(q,\gamma) = \frac{1}{\sqrt{\pi}}ce_n(q,\gamma), \\
\Phi_n^-(q,\gamma) &= \frac{1}{\sqrt{\pi}}se_n(q,\gamma),\ n = 1, 2, ...
\end{aligned} \tag{2.49}$$

They form an orthogonal set. The matrix elements of the gamma depending factors of the transition operator can be easily calculated in Mathematica. Moreover, in the regime of $|q|$-small these matrix elements can be analytically performed, since the following representation of the wave functions hold:

$$\Phi_n^{\pm}(\gamma) \approx \cos(n\gamma - \theta_{\pm}) - \left[\frac{\cos\left[(n+2)\gamma - \theta_{\pm}\right]}{4(n+1)} - \frac{\cos\left[(n-2)\gamma - \theta_{\pm}\right]}{4(n-1)}\right] q^2. \tag{2.50}$$

where $n \geq 3, \theta_+ = 0$ and $\theta_- = \pi/2$. The corresponding energies have very simple expressions:

$$\begin{aligned}
E_0^+ &\approx u - \frac{10}{9}q - \frac{q^2}{2}, \\
E_1^+ &\approx E_1^- \approx u - \frac{10}{9}q - \frac{q^2}{8}, \\
E_2^+ &\approx E_2^- \approx u + 4 - \frac{10}{9}q - \frac{q^2}{2}, \\
E_n^+ &\approx E_n^- \approx u + n^2 - \frac{10}{9}q - \frac{q^2}{2(n^2-1)}, \quad n \geq 3.
\end{aligned} \tag{2.51}$$

Normalizing the above functions to unity, on the γ interval $[-\pi, \pi]$ with respect to the integration measure $d\gamma$ and calculating with the resulting functions the matrix elements of the γ depending factors involved in the electric transition operator one obtains the curves represented in Fig.3.

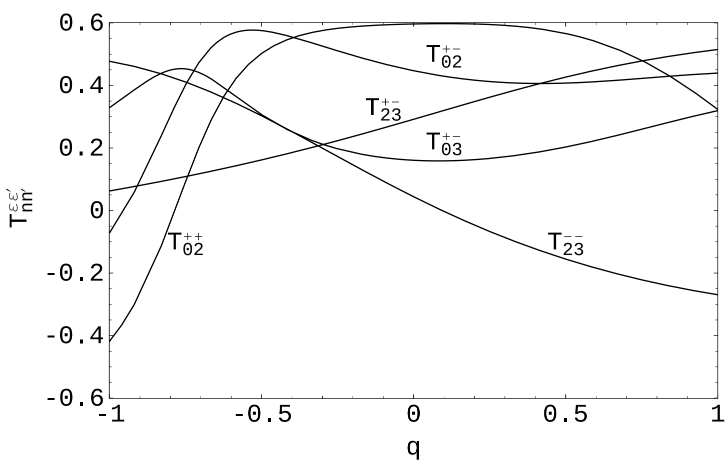


FIG. 3: The matrix elements of $T_{nn'}^{\epsilon\epsilon'}$ for $\cos\gamma$ $(\epsilon\epsilon' = +1)$ and $\sin\gamma$ $(\epsilon\epsilon' = -1$ are represented as functions of q.

Obviously, a phase transition is determined by the combined effects coming from the behavior of the wave function in the β and γ variables, respectively.

For the $X(5)$ symmetry, the β variable is described by a Bessel function of irrational index, while γ by a Laguerre polynomial.

Here we propose to change the description of the γ variable either by a spheroidal or by a Mathieu function. These functions are periodic and the corresponding Hamiltonians Hermitian. Moreover, in both versions, the $X(5)$ Hamiltonian is obtained in the limit of small $|\gamma|$.

D. Including the rotational term preserves periodicity and hermiticity

We recall that so far the rotational term W was left out. Now we turn our attention to this term. If we average W with the Wigner function D^L_{MK} and add the result to the potential U' given by Eq.(2.35) and then following the same path as before, one ends up also with an equation for a spheroidal function. Indeed, let us consider the average of W:

$$\begin{aligned} \langle LK|W|LK\rangle &= \frac{9D}{8\sin^2 3\gamma} - \frac{D - 2K^2 + 2}{8\sin^2\gamma}, \\ D &= L(L+1) - K^2 - 2. \end{aligned} \tag{2.52}$$

When $|\gamma| \ll 1$, this expression admits the following second order expansion in $\sin\gamma$:

$$W(\gamma) = \frac{K^2 - 1}{4\sin^2\gamma} + \frac{1}{3}\Big[L(L+1) - K^2 - 2\Big]\Big(1 + 2\sin^2\gamma\Big). \tag{2.53}$$

The term $L(L+1)/3$ from the above expression, multiplied with the factor $1/\beta^2$ are added to the equation describing the variable β. In the case one makes the option for a infinite well potential in β, the renormalization just mentioned leads to an equation in β, whose solution is a Bessel function with the index

$$\nu = \left(\frac{1}{3}L(L+1) + \frac{9}{4}\right)^{1/2} \tag{2.54}$$

For the case $|\gamma| \ll 1$, we consider the second order expansion in $\sin\gamma$ for the full Hamiltonian (2.28). The result is a trigonometric form for the equation of the spheroidal function:

$$\frac{\partial^2\varphi}{\partial\gamma^2} + \left(E' - \frac{K^2 - 1}{4\sin^2\gamma} - C\sin^2\gamma\right)\varphi = 0, \tag{2.55}$$

with the notations:

$$E' = E + \frac{9}{4} - u_1 - u_2 - \frac{D}{3}, \quad C = \frac{2D}{3} - \frac{9u_1}{2} - 9u_2. \tag{2.56}$$

As suggested by the expression of the starting Hamiltonian, the remaining terms of $W(\gamma)$ should be multiplied with $1/\beta^2$. This coupling of β and γ variables is usually considered as a renormalization term for the potential in γ, by replacing the factor $1/\beta^2$ by the constant $1/\langle\beta^2\rangle$. The notation $\langle\beta^2\rangle$ is used for the expectation value of β^2 in the ground state which results in redefining the constant C, in Eq. (2.55).

Eq. (2.55) can be brought to the form given by Eq.(2.39) by making a successive change of function $S = |\sin\gamma|)^{-1/2}\varphi$ and variable, $x = \cos\gamma$. Indeed, the resulting equation is that of the spheroidal function defined by:

$$\begin{aligned}\lambda_{nm} &= E_{nm} + \frac{7}{2}u_1 + 8u_2 + 2 - D + \frac{1}{3}L(L+1),\\ c^2 &= \frac{9}{2}u_1 + 9u_2 - \frac{2}{3}D.\end{aligned} \tag{2.57}$$

This equation has been used by some authors of this paper, in Ref.[32], to describe the spectrum and the E2 properties of ^{152}Sm.

E. Recovering X(5) in the limit of $|\gamma|$-small

It is worth comparing the present formalism based on the spheroidal functions with the X(5) symmetry method. It is easy to prove that, indeed, the X(5) symmetry is the limiting case for our approach. Indeed, considering the second order expansion in γ of the terms involved in Eq.(2.55) one arrives at:

$$\frac{\partial^2\varphi}{\partial\gamma^2} + \left[E' - \frac{K^2-1}{4}\left(\frac{1}{\gamma^2}+\frac{1}{3}\right) - \left(\frac{K^2-1}{60}+C\right)\gamma^2\right]\varphi = 0. \tag{2.58}$$

This equation is characterizing the $X(5)$ model, with all harmonic contributions included. Indeed, changing the variable $\xi = q\gamma$, with $q = \sqrt{C + \frac{K^2-1}{60}}$, the differential equation becomes:

$$\frac{d^2\varphi}{d\xi^2} + \left[\frac{1}{q^2}\left(E' + \frac{1-K^2}{12}\right) - \xi^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\xi^2}\right]\varphi,\ \alpha = \frac{K}{2}. \tag{2.59}$$

Comparing this equation with that describing an harmonic, isotropic plane oscillator:

$$\frac{d^2\Phi_{n\alpha}}{d\xi^2} + \left[2(n+\alpha+1) - \xi^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\xi^2}\right]\Phi_{n\alpha} = 0, \quad \alpha = \frac{K}{2}. \tag{2.60}$$

one identifies the function φ with the function Laguerre:

$$\varphi_{n\alpha} = \sqrt{2\frac{n!}{\Gamma(n+\alpha+1)}}\xi^\alpha L_n^\alpha(\xi^2)\exp\left(-\frac{1}{2}\xi^2\right), \tag{2.61}$$

while the system energy is:

$$E_n = 2\left(C + \frac{K^2-1}{60}\right)^2\left(n + \frac{K}{2} + 1\right) + u_1 + u_2 + \frac{1}{4}K^2 - 3. \tag{2.62}$$

The property of reaching the $X(5)$ symmetry in the limit of small values of γ, holds also for the Mathieu functions. Indeed, these functions satisfy a differential equation which is of spheroidal type. Consequently, in the limit $|\gamma| \ll 1$, the Mathieu functions describe the irreducible representations of the group $X(5)$. Numerical applications with Mathieu functions will be published elsewhere.

III. THE PRESENT APPROACH

Here we summarize the procedure adopted in the present paper to treat the variables β and γ.

The potential in the two variables is considered to be of the form given by Eq. (2.2). As $V(\beta)$ we take the Davidson potential (2.14). Including the terms proportional to $\frac{1}{\beta^2}$ (this is $\frac{1}{3\beta^2}\left[L(L+1)\right]$) from the rotational term in the Schrödinger equation for β , the resulting equation admits solutions which formally coincide with those given by Eqs. (2.8) and (2.9) but having instead of τ a irrational quantum number p defined as:

$$p = -\frac{3}{2} + \left[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4\right]^{1/2} \tag{3.1}$$

This expression is obtained by writing the coefficient of $\frac{1}{\beta^2}$ from the Schrödinger equation associated to the β variable in the form:

$$(p+1)(p+2) = 2 + \frac{1}{3}L(L+1) + \beta_0^4. \tag{3.2}$$

This equation has two solutions, one written above, while the second one is differing from the first one by the sign of the square root term. Let us denote for a while the two solutions by $p_{\pm}$. Note that the Davidson potential is not a continuous function in $\beta = 0$. This causes the fact that for L=0, we have:

$$\lim_{\beta_0\to 0} p_+ = 0, \quad \lim_{\beta_0\to 0} p_- = -3. \tag{3.3}$$

Consequently the corresponding spectra are given by:

$$\begin{aligned} E_n^{(+)} &= 2n + \frac{5}{2}, \\ E_n^{(-)} &= 2n - 3 + \frac{5}{2}. \end{aligned} \tag{3.4}$$

Therefore the full spectrum of the 5-dimensional oscillator is recovered only if both solutions are considered at a time [52]. Since, as we shall see a bit later, β_0 is far from origin we make the option for the branch corresponding to the eigenvalue p_+. The choice is justified by the fact that for $\beta_0 \neq 0$ the wave function corresponding to p_- is singular in origin and therefore it is not a convenient solution.

As explained before, for the states belonging to the ground band β_0 was fixed variationally, by Eq.(2.17). We extended Eq. (2.17) to the beta and gamma bands, respectively. The first derivatives of the ratios $R_L^{(k)}$, defined by

$$\begin{aligned} R_L^{(\beta)} &= \frac{E_{L_\beta^+} - E_{0_\beta^+}}{E_{2_\beta^+} - E_{0_\beta^+}}, L \geq 4, \\ R_L^{(\gamma)} &= \frac{E_{L_\gamma^+} - E_{2_\gamma^+}}{E_{3_\gamma^+} - E_{2_\gamma^+}}, L \geq 4, \end{aligned} \tag{3.5}$$

have the β_0 dependence shown in Fig.4 for some particular values of$L$. We fix β_0 for the states in the band k (=β, γ) so that the first derivatives of the ratios $R_L^{(k)}$ are maximum. From Fig. 4 one sees that each curve has a well pronounced maximum. Collecting the values of β_0 obtained in this way, and representing them as function of L one obtains a straight line for both beta and gamma bands, as shown in Fig.5. Extrapolating these straight lines for L=0,2 in beta band and L=2,3 in the gamma band, one obtains a one to one correspondence between the states in the two bands and the values of β_0.

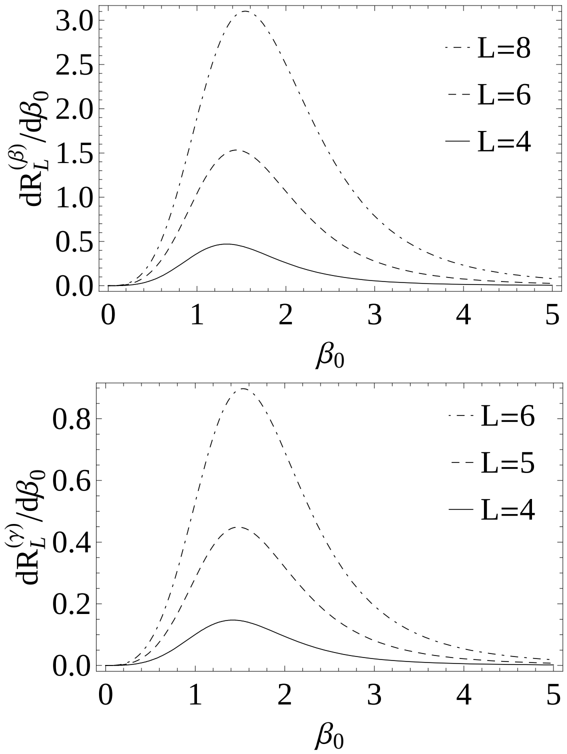


FIG. 4: The first derivative of the ratios $R_L^{(\beta)}$ (upper panel) and $R_L^{(\gamma)}$ (lower panel) are plotted as function of β_0.

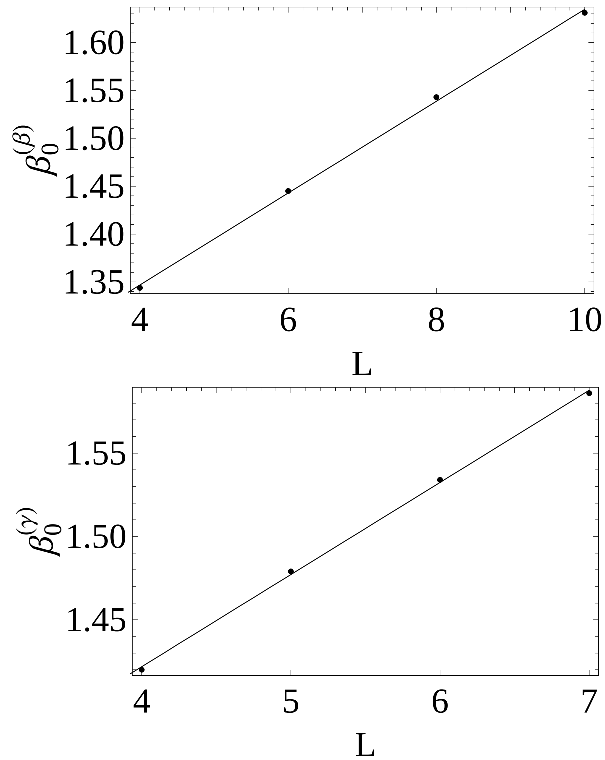


FIG. 5: The solutions $\beta_L^{(k)}$ of the equation $\frac{d^2R^{(k)}}{d\beta_0^2} = 0$ for $k = \beta$ (upper panel) and $k = \gamma$ (lower panel), are plotted as functions of L.

Let us denote by $E_{np}^{(\beta,D)}(=2n+p+\frac{5}{2})$ the energy provided by the Shrödinger equation associated to the variable beta. The index D suggests that the β potential is chosen to be of Davidson type.

For comparison we considered also an infinite square well potential for the β variable. In this case the energies associated to the beta variable are denoted by $E_{\xi s}^{(\beta,B)}$. They correspond to the Bessel function of index $s+3/2$ with s defined by the following equation:

$$\left(s+\frac{3}{2}\right)^2=\frac{9}{4}+\frac{1}{3}L(L+1) \tag{3.6}$$

ξ is an ordering index for the zeros of the Bessel function. Therefore the irrational index for the Bessel function will be:

$$s=-\frac{3}{2}+\sqrt{\frac{1}{3}L(L+1)+\frac{9}{4}}. \tag{3.7}$$

Other possible ways of renormalizing the differential equation for β are discussed in Appendix C. The cases when the spheroidal function formalism for the γ variable is considered at a time with oscillator potential or hybrid potential potential in β will discussed elsewhere.

As for the potential in γ we considered

$$U(\gamma)=\frac{1}{\langle\beta^2\rangle}\left[u_1\cos 3\gamma+u_2\cos^2 3\gamma+\frac{9}{4\sin^2 3\gamma}\right]. \tag{3.8}$$

Assuming that $|\gamma|\ll 1$ the rotational term is expanded in powers of $\sin 3\gamma$. From the rotational term we depict the term not depending on γ and proportional to $L(L+1)$, otherwise being proportional to $\frac{1}{\beta^2}$, and add it to the Hamiltonian in beta which results in having a renormalization of the centrifugal term. In the remaining terms we approximate $1/\beta^2$ by $1/\langle\beta^2\rangle$. Thus, the Hamiltonian in γ will comprise an overall factor $1/\langle\beta^2\rangle$. If in the Hamiltonian which multiplies this factor, one changes the function $\varphi\rightarrow S=|\sin 3\gamma|^{-1/2}\varphi$ and the variable $\gamma\rightarrow x=\cos 3\gamma$, the corresponding Schrödinger equation is that of a spheroidal function defined by Eq. (2.39) with

$$\begin{aligned}
\lambda_{nm} &= \frac{1}{9}\left(E_{nm}^{(\gamma)}-\frac{1}{2}u_1-\frac{11}{27}D+\frac{1}{3}L(L+1)\right),\\
c^2 &= \frac{1}{9}\left(\frac{1}{2}u_1+u_2-\frac{2}{27}D\right),\\
m &= \frac{K}{2}.
\end{aligned} \tag{3.9}$$

For illustration, in Figs. 6, 7 we give few potentials corresponding to $\langle\beta\rangle = 1$ and different sets of (u_1, u_2). The spheroidal functions normalized to unity in the interval $[0, \pi/3]$, given by Eq. (2.39) with the parameter c^2 determined by (u_1, u_2) used in Figs 6,7 are represented as functions of γ in Figs. 8, 9 for three L levels from the ground band. Once we fix the γ potential, we can calculate the energy associated to the γ variable.

The total energy for the system described by the decoupled β and γ variables is:

$$E^{(k)}_{n\tau;n'm;LK} = E_0 + AE^{(\beta,k)}_{n\tau} + BE^{(\gamma)}_{n'm}, \quad k = D, B. \tag{3.10}$$

Note that the energy determined by the rotational degrees of freedom has been already included when the term of the Hamiltonian denoted by W was averaged with the Wigner functions.

If the variable β is described by a Schrödinger equation with the Davidson potential, the wave function describing the whole system is:

$$|np; n'm; LMK\rangle = \Psi_{np}(\beta)S_{n'm}(\cos 3\gamma)\frac{\sqrt{2L+1}}{4\pi}\left(D^L_{MK} + (-1)^{L+K}D^L_{M,-K}\right), m = \frac{K}{2}. \tag{3.11}$$

The ground, beta and gamma bands are defined by the quantum numbers:

$$\begin{aligned} n &= 0,\ n' = 0,\ m = 0, K = 0, L = 0, 2, ... \text{ ground band,} \\ n &= 0,\ n' = 1,\ m = 1, K = 2, L = 2, 3, ... \text{ gamma band,} \\ n &= 1,\ n' = 0,\ m = 0, K = 0, L = 0, 2, ... \text{ beta band.} \end{aligned} \tag{3.12}$$

In the situation when the β potential is an infinite square well, the wave function has the expression:

$$|ns; n'm\rangle = f_{\xi s}(\beta)S_{n'm}(\cos 3\gamma)\frac{\sqrt{2L+1}}{4\pi}\left(D^L_{MK} + (-1)^{L+K}D^L_{M,-K}\right), m = \frac{K}{2}. \tag{3.13}$$

with $f_{\xi\tau}$ given by Eq.(2.10) and the irrational index s given by Eq.(3.6) and $S_{n'm}(\cos 3\gamma)$ defined by Eq.(2.39). The quantum numbers defining the ground, beta and gamma bands are as follows:

$$\begin{aligned} \xi &= 1,\ n' = 0,\ m = 0, K = 0, L = 0, 2, ... \text{ ground band,} \\ \xi &= 1,\ , n' = 1,\ m = 1, K = 2, L = 2, 3, ... \text{ gamma band,} \\ \xi &= 2,\ , n' = 0,\ m = 0, K = 0, L = 0, 2, ... \text{ beta band.} \end{aligned} \tag{3.14}$$

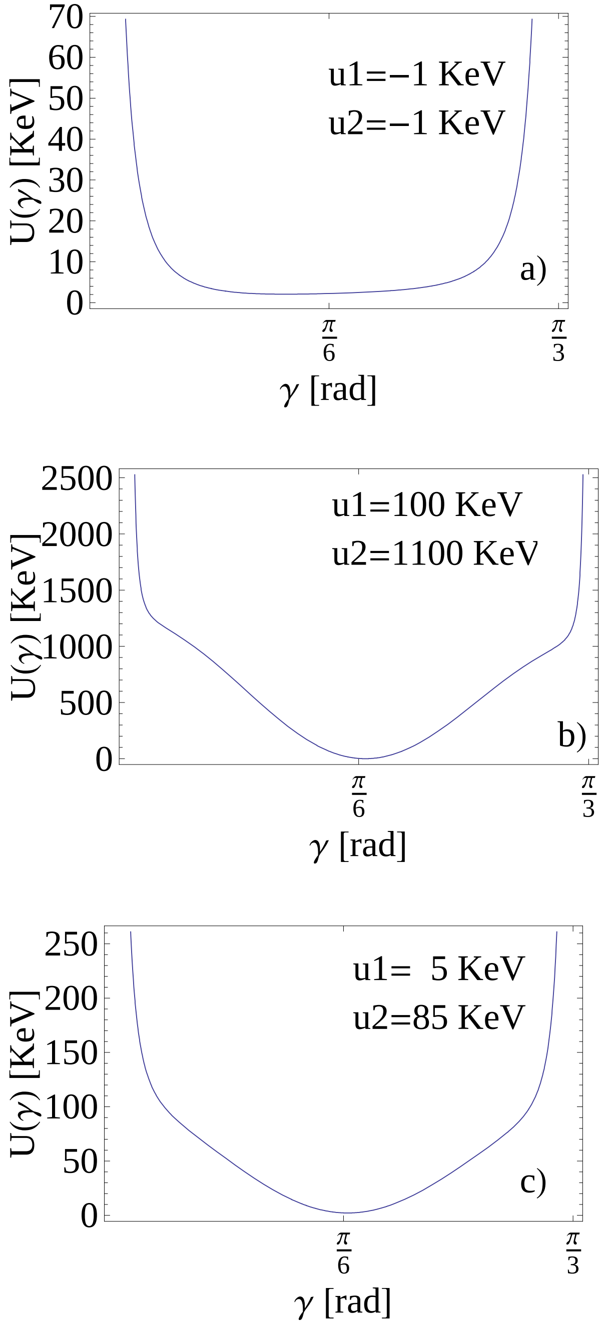
.

Once the wave functions are determined by solving the eigenvalue equations for β and γ, we can proceed to calculating the electric transition probabilities. In order to get a feeling about how sensitive the matrix elements of γ depending terms of the transition operator are to changing the parameter c, we have plotted them in Figs.10, 11, versus c. From there one notices that in a large interval of c, the matrix elements are slowly varying with c. The diagonal matrix elements of $\cos\gamma$ (the first panel in the left column) are slightly increasing by 0.01 starting with the values 0.823, 0.844 and 0.854 at $c = 0$. The corresponding matrix elements of $\sin\gamma$ are changing just a little when we vary c, starting with the values 0.475 0.487 and 0.493. The magnitudes of the matrix elements between states belonging to the same multiplet (see Fig.1) are small. The matrix elements characterized by the same $\Delta n = 1$ are relatively large for $\cos\gamma$ but small for $\sin\gamma$.

The reduced E2 transition probabilities have been calculated by using alternatively a harmonic, $T^{(h)}$, and an anharmonic transition operator, $T^{(anh)}_{2\mu}$, having the expressions:

$$
\begin{aligned}
T^{(h)}_{2\mu} &= t\beta\left(\cos\gamma D^2_{\mu 0} + \frac{\sin\gamma}{\sqrt{2}}(D^2_{\mu 2} + D^2_{\mu,-2})\right),\\
T^{(anh)}_{2\mu} &= t_1\beta\left(\cos\gamma D^2_{\mu 0} + \frac{\sin\gamma}{\sqrt{2}}(D^2_{\mu 2} + D^2_{\mu,-2})\right) + \\
&\qquad t_2\sqrt{\frac{2}{7}}\beta^2\left(-\cos 2\gamma D^2_{\mu 0} + \frac{\sin 2\gamma}{\sqrt{2}}(D^2_{\mu 2} + D^2_{\mu,-2})\right)
\end{aligned}
\tag{3.15}
$$

The strengths t, t_1 and t_2 are free parameters which are fixed by fitting one and two particular $B(E2)$ values, respectively. Due to the structure of the wave functions specified above, the matrix elements between the states involved in a given transition are factorized into matrix elements of the transition operators factors depending on β, γ and the Euler angles, respectively.



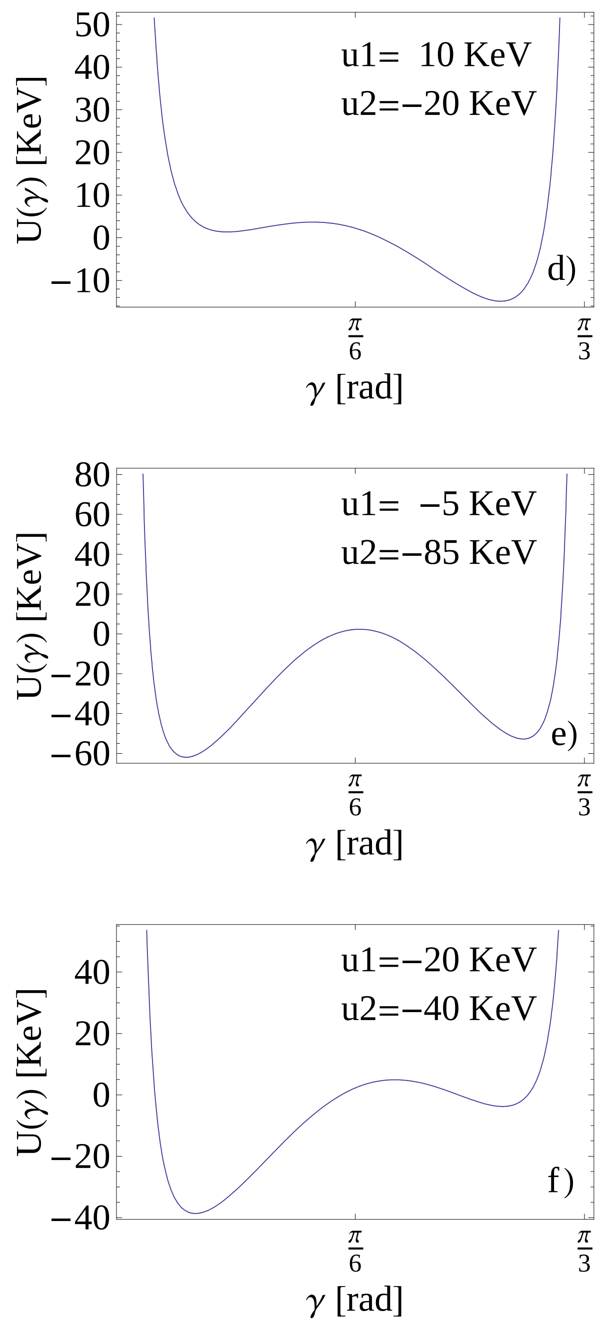
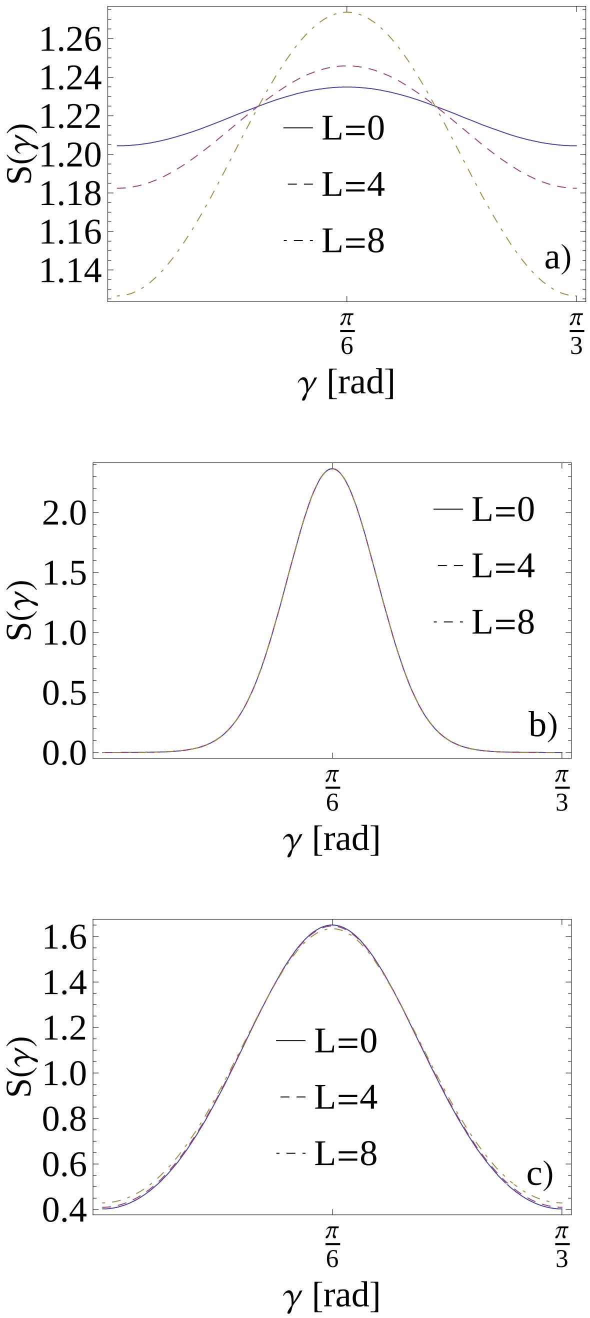


FIG. 6: The γ potential for different values of the parameters u_1 and u_2.

FIG. 7: The γ potential for different values of the parameters u_1 and u_2.



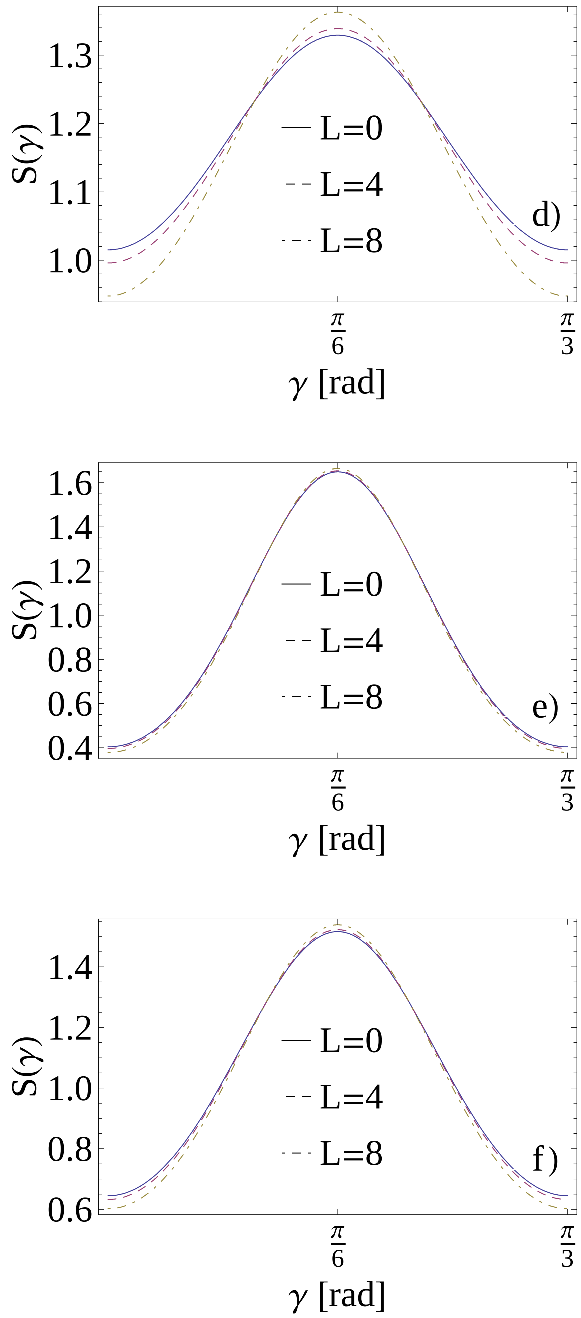
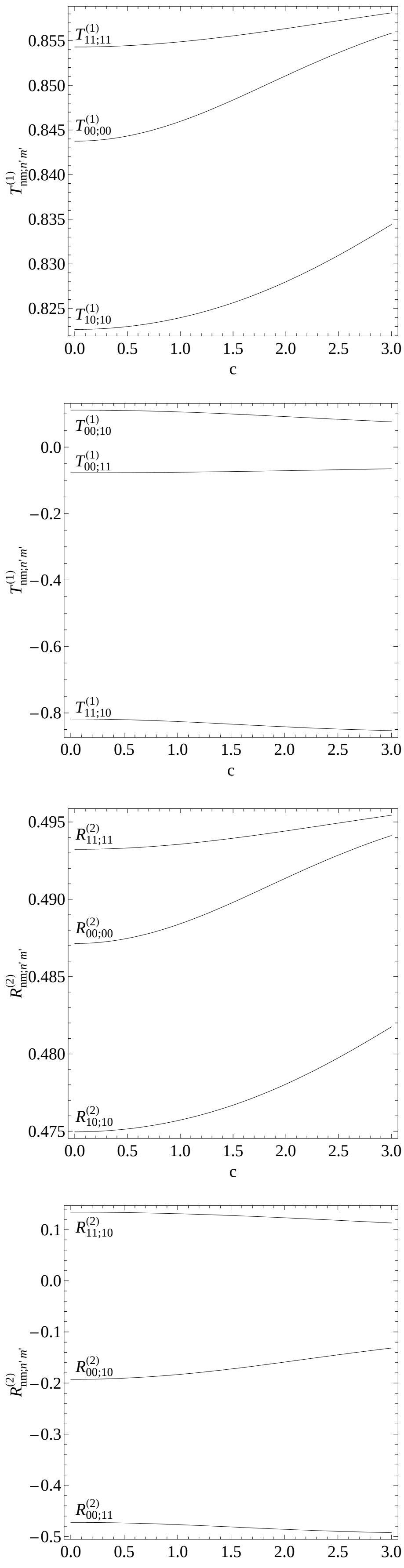


FIG. 8: The spheroidal functions corresponding to the γ potentials from Figs. 4 a), b) and c) respectively, are plotted versus γ. The functions describe the ground band states of angular momenta 0, 2 and 4, respectively.

FIG. 9: The spheroidal functions corresponding to the γ potentials from Figs. 4 d), e) and f) respectively, are represented versus γ. The functions describe the ground band states of angular momenta 0, 2 and 4 respectively.



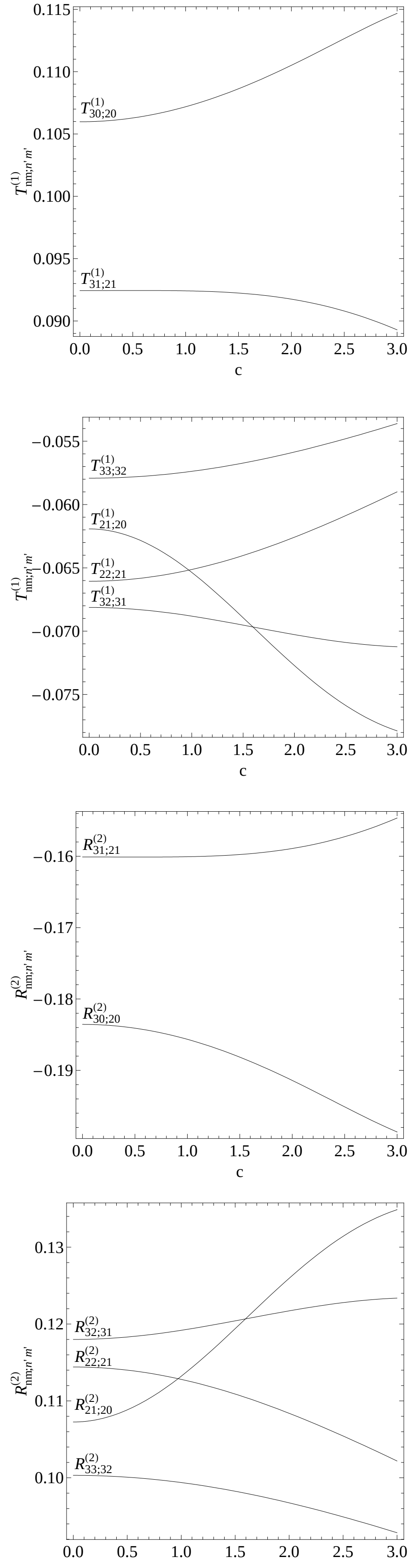


FIG. 10: The matrix elements of $\cos\gamma$ ($T^{(1)}_{nm;n'm'}$) and $\sin\gamma$ ($R^{(2)}_{nm;n'm'}$) are plotted as functions of the parameter c involved in the

FIG. 11: The same as in Fig.(10) but for different spheroidal functions.

IV. NUMERICAL RESULTS

The formalism described in the previous Section has been applied for ^{150}Nd, ^{154}Gd and ^{192}Os. The choice is justified by the values of the ratio of the excitation energies for the first two excited states in the ground band. Indeed, these are 2.93, 3.015 and 2.82 respectively and therefore they are expected to have the features of X(5) symmetry. As we stated in Introduction the present formalism is close to the X(5) symmetry. Indeed, it goes to $X(5)$ in the limit $\gamma \rightarrow 0$ and, on the other hand, the spheroidal function equation has been derived by expanding the gamma depending terms in the initial Hamiltonian in terms of $\sin 3\gamma$. However, it brings two new things namely i) the wave function is periodic in γ, the matrix elements of γ depending functions being performed with the measure $|\sin 3\gamma|$ and ii) the factor function describing the β degree of freedom satisfy a Schrödinger equation involving the Davidson potential. Numerical calculations will show us what are the quantitative corrections to the $X(5)$ picture brought by curing the drawbacks of the preceding approaches.

The γ potentials for the three nuclei have been chosen from those given in Fig.6 Indeed, the potentials ^{154}Gd, ^{150}Nd and ^{192}Os are those from Figs. 6 a), c), and b) respectively. The spheroidal functions corresponding to these potentials are represented in Figs 8 a), 8 c) and 8 b), respectively. We varied the shapes of gamma potential and made the option for that one which yields a good agreement between the calculated B(E2) values associated to the transitions from gamma to ground band and the corresponding experimental data.

The energy levels in the three bands are given in units of $E_{2_g^+}$ and therefore we need only the ratio of the parameters A and B involved in the energy expression given by Eq. (3.10). The term E_0 is not depending on the quantum number defining the states but it is considered to depend on band. In our calculation E_0 for ground and beta bands are taken to be equal and therefore they do not affect the relative energies in the two bands, while for gamma band was fixed so that the head state energy is recovered. The parameters defining the transition operator have been fixed by fitting the B(E2) values for the transitions $2_g^+ \rightarrow 0_g^+$ and $2_\gamma^+ \rightarrow 0_g^+$. The results obtained in this manner are collected in Table I.

Now let us proceed at describing separately the results for each of the three nuclei considered here.

The calculated energies for ^{150}Nd are listed in Table II. The results of present paper are given in the columns D and ISW. They have been obtained by using a spheroidal

	^{150}Nd	^{154}Gd	^{192}Os
$u_1[keV]$	100	-1	5
$u_2[keV]$	1100	-1	85
$E_0[keV]$	-27.696	-222.243	9.635
	-3.545	-12.892	0.367
B/A	0.532	17.728	$0.191 \cdot 10^{-3}$
	0.0328	1.027	$2.896 \cdot 10^{-3}$
$t[efm^2]$	147.889	124.065	90.583
	161.909	135.831	99.174
$t_1[efm^2]$	209.443	184.933	120.976
	228.706	219.473	138.962
$t_2[efm^2]$	100.178	106.240	50.077
	116.624	156.629	71.225

TABLE I: The parameters $u_1, u_2, E_0, B/A$, involved in the energy expression (2.9), calculated by the method described in the text, are given for ^{150}Nd, ^{154}Gd, ^{192}Os. Also we give the values for the parameters t and t_1, t_2 involved in the harmonic and anharmonic transition operators, respectively. They were obtained by fiting the B(E2) values for the transition $2_g^+ \rightarrow 0_g^+$ if the harmonic transition operator is used, and for the transitions $2_g^+ \rightarrow 0_g^+$ and $2_\gamma^+ \rightarrow 0_g^+$ for the anharmonic transition operator.

function description for γ variable while for β an infinite square well potential (SWP) and the Davidson's potential (D), have been alternatively used.These results are compared with the corresponding experimental data [37, 38], listed in the column headed by $Exp.$, as well as with the theoretical results obtained within the $X(5)$ symmetry approach. If we enlarge the list for each band the deviations of predictions from the experimental data are increasing functions of angular momentum.

State	Exp.	X(5)	ISW	D
2_g^+	1	1	1	1
4_g^+	2.93	2.90	2.93	2.93
6_g^+	5.53	5.43	5.53	5.53
8_g^+	8.68	8.48	8.70	8.73
10_g^+	12.28	12.03	12.42	12.50
12_g^+	16.27	16.04	16.66	16.82
0_β^+	5.19	5.65	5.30	4.20
2_β^+	6.53	7.45	7.05	5.45
4_β^+	8.74	10.69	10.24	7.65
6_β^+	11.84	14.75	14.27	10.64
2_γ^+	8.16		8.16	8.16
3_γ^+	9.22		9.03	9.35
4_γ^+	10.39		10.08	10.70

TABLE II: Excitation energies of some states from ground, beta and gamma bands of ^{150}Nd, given in units of the 2_g^+ energy, obtained with three differents approaches, $X(5)$, infinite square well potential (ISW) for β and spheroidal functions formalism for γ, Davidson's β potential plus spheroidal functions method for γ (D), are compared with the corresponding experimental data.

The intraband ground band and beta band transitions as well the gamma to ground and beta to ground transtions were calculated by using both a harmonic and a anharmonic structure for the transtion operators with the strength parameters from Table I.The final results are those from Table III. In the mentioned Table we give also the corresponding experimental data, taken from Ref.[37, 38], as well as the results provided by the X(5) formalism. It is interesting to remark that except for the intraband gamma to ground transitions, the harmonic approach of the ISW calculations provides identical results with the X(5) calculations. We may say that the agreement with the data is improved by adding the anharmonic effects for some transitions but for some other the discrepancies are increased. In an overall analysis, the agreement is improved by anharmonicities.

$B(E2;J^+_k \to J'^+_{k'}$	Exp.	X(5)	ISW		D	
$2^+_g \to 0^+_g$	115	115	115	115	115	115
$4^+_g \to 2^+_g$	182	184	184	177	197	182
$6^+_g \to 4^+_g$	210	228	228	210	266	226
$8^+_g \to 6^+_g$	278	262	262	233	334	260
$10^+_g \to 8^+_g$	204	288	289	249	394	387
$2^+_\beta \to 0^+_\beta$	114	92	92	91	148	121
$4^+_\beta \to 2^+_\beta$	170	138	138	136	210	167
$0^+_\beta \to 2^+_g$	39	72	72	43	105	58
$2^+_\beta \to 0^+_g$	1.2	2.4	2.4	0.4	0.6	0.1
$2^+_\beta \to 2^+_g$	9	10	10	4	9	3
$2^+_\beta \to 4^+_g$	17	42	42	26	69	36
$4^+_\beta \to 2^+_g$	0.12	0.56	1	0.01	0.61	4.6
$4^+_\beta \to 4^+_g$	7	7	7	3	4	0.3
$4^+_\beta \to 6^+_g$	70	32	32	18	51	22
$2^+_\gamma \to 0^+_g$	3.0	3.4	0.6	3	0.6	3
$2^+_\gamma \to 2^+_g$	5.4	5.1	0.9	4.7	1	5.0
$2^+_\gamma \to 4^+_g$	2.6	0.3	0.05	0.3	0.05	0.3
$4^+_\gamma \to 2^+_g$	0.9	2.3	0.4	2.1	0.4	2.3
$4^+_\gamma \to 4^+_g$	3.9	7.3	1.3	6.8	1.5	8.2

TABLE III: The B(E2) values for ^{150}Nd calculated in three different formalisms, X(5),ISW,B, and the corresponding experimental data are given in units of $e^2fm^4 \times 10^2$. The results of the present work are obtained by using spheroidal functions for γ and alternatively an infinite square well potential (ISW) and Davidson's potential (D) for the variable β. The results for $X(5)$ formalism are taken from Ref.[37]. In the first columns headed by ISW and D respectively, are the results obtained with a harmonic quadrupoole transition operator. In the second columns ISW and D we give the results obtained with an anharmonic quadrupole transition operator. Calculations made with the $X(5)$ formalism have used a harmonic transition operator

State	Exp.	X(5)	ISW	D	CSM	CSM2
2_g^+	1	1	1	1	1	1
4_g^+	3.015	2.90	3.015	3.015	3.110	2.932
6_g^+	5.83	5.43	5.84	5.83	6.105	5.612
8_g^+	9.30	8.48	9.39	9.35	9.835	8.936
10_g^+	13.30	12.03	13.55	13.81	14.188	12.850
0_β^+	5.53	5.65	4.12	3.35	5.662	5.335
2_β^+	6.63	7.45	5.71	4.51	6.413	6.025
4_β^+	8.51	10.69	8.70	6.68	8.101	7.714
6_β^+	11.10	14.75	12.63	9.75	10.626	10.325
8_β^+	14.27	19.44	17.36	13.58	13.879	13.756
10_β^+	17.83	24.69	22.79	18.15	17.782	17.916
2_γ^+	8.10		8.10	8.10	7.771	7.817
3_γ^+	9.16		9.00	8.89	9.047	8.528
4_γ^+	10.27		9.87	10.33	10.073	9.514
5_γ^+	11.64		11.01	11.74	11.301	10.604
6_γ^+	13.05		12.31	13.31	12.778	12.002
7_γ^+	14.71		13.75	15.02	14.386	13.421

TABLE IV: The same as in Table II but for ^{154}Gd.

Although ^{154}Gd is a deformed nucleus, having the quadrupole deformation ≈ 0.25 [46], the authors of Ref.[39] consider it as a good candidate for the critical point in a phase transition which takes place along the chain of isotopes. This shape transition has been studied recently by two of us (A.A.R. and A. F.) in Ref.[43] within the Coherent State Model (CSM). Here the results obtained through ISW and D formalism are compared to those obtained within other phenomenological approaches like X(5), CSM, and CSM2. The CSM2 differs from CSM by the model states for the beta band. A full list of references concerning CSM may be found in Ref.[9]. The results for X(5) are taken from Ref.[47] with a suitable scaling when one passes from ^{152}Sm to ^{154}Gd. We notice that the D formalism provides energies which are closest to the experimental data[40–42]. Concerning the B(E2) values, from Table V one notices that B and ISW formalisms are reproducing better the data [40, 41] for the intraband ground and beta bands transitions. As regard the beta to ground transitions the X(5) formalism yields an overall better agrement with the data. Also, we note that the approach D is describing quite well the transitions $J^+_\beta \to (J+2)^+_g$ but its predictions for $J^+_\beta \to J^+_g$ are smaller than the corresponding data by a factor of about 10. By contrary the theoretical transitions $J^+_\beta \to (J-2)^+_g$ are larger by a factor 10 to 50 than the experimental results. The best description for the interband transitions collected in Table V is provided by the ISW approach. The anharmonic term of the transition operator brings important contribution to both the intra and interband transitions,.

Transition	Exp.	X(5)	ISW		D		CSM		CSM2	
$2_g^+ \to 0_g^+$	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3	77.3
$4_g^+ \to 2_g^+$	117.8	123.5	123.7	117.8	132.4	117.8	114.4	114.9	117.2	117.3
$6_g^+ \to 4_g^+$	138.2	153	153.4	138.9	179.1	138.9	133.1	133.9	139.7	139.9
$8_g^+ \to 6_g^+$	152.6	176.1	176.3	152.8	225	152.2	148.7	148.4	159.4	159.8
$10_g^+ \to 8_g^+$	173.1	193.8	194.6	162.4	265.6	160	163.7	163.8	178.4	178.9
$2_\beta^+ \to 0_\beta^+$	49	61.7	61.5	61.3	99.2	72.7	103.8	67.1	102.8	103.8
$4_\beta^+ \to 2_\beta^+$	122	92.9	92.8	91.2	141.1	99.3	152.4	99.5	150.9	152.4
$6_\beta^+ \to 4_\beta^+$	111	113.6	113.6	109.5	152.4	107.5	175.6	116.2	174	175.6
$0_\beta^+ \to 2_g^+$	25.8	48.8	48.2	26	70.6	26.5	1(-3)	1.32	19.9	20.0
$2_\beta^+ \to 2_g^+$	4.0	7.0	6.4	2.1	6.1	0.44	2(-4)	0.34	4.13	4.16
$2_\beta^+ \to 4_g^+$	11.9	27.9	28.3	15.4	46.4	15.9	3(-3)	0.76	12.98	13.08
$4_\beta^+ \to 2_g^+$	0.35	0.54	0.73	0.06	0.4	5.4	0	0.21	2.17	2.19
$4_\beta^+ \to 4_g^+$	3.8	4.83	4.72	1.29	2.8	0.044	3(-3)	0.33	3.44	3.47
$4_\beta^+ \to 6_g^+$	12	21.5	21.6	10.7	34.3	8.16	8(-3)	0.68	13.42	13.53
$6_\beta^+ \to 4_g^+$	0.27	0.50	0.50	0.18	7	17.3	1(-4)	0.17	1.70	1.71

TABLE V: The same as in Table II but for ^{154}Gd. The notation $k(-m)$ stands for the number $k \cdot 10^{-m}$.

$L_i, L_f; L_i^{'}, Lf^{'}$	Exp.	X(5)	ISW		D		CSM		CSM2	
$4_g, 2_g; 2_g, 0_g$	1.52	1.60	1.60	1.52	1.71	1.52	1.48	1.48	1.52	1.52
$6_g, 4_g; 4_g, 2_g$	1.17	1.24	1.24	1.18	1.35	1.18	1.16	1.16	1.19	1.19
$8_g, 6_g; 6_g, 4_g$	1.10	1.15	1.15	1.10	1.26	1.10	1.12	1.11	1.14	1.14
$10_g, 8_g; 8_g, 6_g$	1.13	1.10	1.10	1.06	1.18	1.06	1.10	1.10	1.119	1.113
$2_\gamma, 0_g; 2_\gamma, 2_g$	0.468	0.666	0.654	0.642	0.618	0.594	0.509	0.468	0.501	0.468
$2_\gamma, 4_g; 2_\gamma, 2_g$	0.144	0.052	0.053	0.054	0.054	0.057	0.087	0.131	0.068	0.069
$3_\gamma, 2_g; 3_\gamma, 4_g$	1.006	2.368	2.335	2.274	2.208	2.096	1.302	0.975	1.432	1.289
$4_\gamma, 2_g; 4_\gamma, 4_g$	0.148	0.315	0.311	0.304	0.286	0.273	0.159	0.126	0.152	0.123
$4_\gamma, 6_g; 4_\gamma, 4_g$	0.27	0.088	0.090	0.092	0.092	0.097	0.377	0.377	0.115	0.117
$5_\gamma, 4_g; 5_\gamma, 6_g$	0.744	1.667	1.655	1.619	1.560	1.486	0.657	0.401	0.786	0.665
$6_\gamma, 4_g; 6_\gamma, 6_g$	0.081	2.333	0.25	0.246	0.229	0.219	0.081	0.046	0.076	0.050
$2_\gamma, 2_\beta; 2_\gamma, 2_g$	1.00	0.032	0.054	0.108	0.061	0.145	1.206	0.751	2.238	0.048
$2_\beta, 0_g; 2_\beta, 2_g$	0.123	0.429	0.257	0.067	0.069	1.716	0.0	0.561	0.538	0.475
$2_\beta, 4_g; 2_\beta, 2_g$	2.76	4.00	4.45	7.20	7.6	36	13.24	2.257	3.141	3.141
$4_\beta, 2_g; 4_\beta, 4_g$	0.086	0.112	0.155	0.046	0.145	121	0.001	0.625	0.630	0.630
$4_\beta, 6_g; 4_\beta, 4_g$	2.63	4.45	4.57	8.30	12	184	2.484	2.073	3.896	3.896
$6_\beta, 4_g; 6_\beta, 6_g$	0.08		0.13	0.22	2475	292	0.0	0.071	0.555	0.555
$2_\beta, 0_g; 2_\beta, 0_\beta$	0.008	0.049	0.027	0.002	0.004	0.568	0.0	0.0028	0.022	0.019
$4_\beta, 2_g; 4_\beta, 2_\beta$	0.0025	0.0058	0.0079	0.0006	0.0029	0.530	0.0	0.0021	0.014	0.014
$6_\beta, 4_g; 6_\beta, 4_\beta$	0.0024	0.0044	0.0044	0.0017	0.045	0.676	0.0	0.0015	0.010	0.010
$8_\beta, 6_g; 8_\beta, 6_\beta$	0.006		0.003	0.002	0.204	2.147	0.977	0.0011	0.007	0.007

TABLE VI: Calculated branching ratios $B(E2; L_i^+ \rightarrow L_f^+)/B(E2; L_i^{'+} \rightarrow L_f^{'+})$ (denoted by $L_i, L_f; L_i^{'}, Lf^{'}$)for some interband as well intraband transitions in ^{154}Gd are compared with the corresponding experimental data. The results of this paper, labeled by ISW and D, were obtained by using spheroidal functions for γ and for the β variable the infinite square well and Davidson potentials, respectively. Results from the first columns headed by ISW and D were obtained by using a harmonic transition quadrupole operator while in the second columns are listed the results of our calculations with a anharmonic transition operator.

Starea	Exp.	X(5)	ISW	D
2_g^+	1	1	1	1
4_g^+	2.82	2.90	2.90	2.90
6_g^+	5.29	5.43	5.43	5.43
8_g^+	8.30	8.48	8.48	8.48
10_g^+	11.75	12.03	12.03	12.01
12_g^+	15.60	16.04	16.04	16.01
0_β^+	4.65	5.65	5.65	4.44
2_β^+	5.48	7.45	7.45	5.73
2_γ^+	2.38		2.38	2.38
3_γ^+	3.35		3.24	3.32
4_γ^+	4.42		4.28	4.44
5_γ^+	5.56		5.48	5.75
6_γ^+	7.12		6.81	7.20
7_γ^+	8.32		8.27	8.81
8_γ^+	10.37		9.86	10.54
10_γ^+	14.06		13.41	14.40

TABLE VII: The same as in Table II but for ^{192}Os

Some branching ratios of the intraground band as well as of the interband transitions are given in Table VI.The failure of the D approach to describe these branchings for the $\beta \rightarrow g$ transitions is noticeable. The reason consists in the fact that within this formalism the predictions for the transitions $J_\beta^+ \rightarrow J_g^+$ are too small comparing them with the experimental data. The other ratios are described reasonable well by all theoretical models.

The results for ^{192}Os are presented in Tables VII and VIII. Therein we give also the experimental data and the theoretical results yielded by the $X(5)$ formalism. From Table VII we remark the very good description of energy levels by the D formalism. Concerning the E2 transitions the data from Table VIII show that the ISW formalism with a harmonic transition operator yields results very close to the ones produced with the $X(5)$ formalism.

Concluding the numerical analysis of this Section we may say that the formalism presented

Transition	Exp.	X(5)	ISW		D	
$2_g^+ \to 0_g^+$	0.424	0.424	0.424	0.424	0.424	0.424
$4_g^+ \to 2_g^+$	0.497	0.678	0.678	0.656	0.726	0.673
$6_g^+ \to 4_g^+$	0.660	0.841	0.840	0.787	0.981	0.836
$8_g^+ \to 6_g^+$	0.754	0.966	0.964	0.879	1.23	0.966
$10_g^+ \to 8_g^+$	0.688	1.060	1.06	0.947	1.45	1.06
$4_\gamma^+ \to 2_\gamma^+$	0.298	0.269	0.275	0.274	0.302	0.288
$6_\gamma^+ \to 4_\gamma^+$	0.336	0.611	0.610	0.590	0.730	0.643
$8_\gamma^+ \to 6_\gamma^+$	0.314	0.823	0.807	0.761	1.05	0.858
$2_\gamma^+ \to 0_g^+$	0.037	0.012	0.009	0.037	0.009	0.037
$2_\gamma^+ \to 2_g^+$	0.303	0.019	0.014	0.057	0.015	0.062
$4_g^+ \to 2_\gamma^+$	0.014	0.001	0.006	0.026	0.007	0.029
$4_\gamma^+ \to 2_g^+$	0.002	0.008	0.006	0.026	0.007	0.028
$4_\gamma^+ \to 4_g^+$	0.203	0.027	0.020	0.084	0.023	0.102
$6_g^+ \to 4_\gamma^+$	0.012	0.006	0.006	0.025	0.007	0.032
$6_\gamma^+ \to 4_g^+$	0.0004	0.006	0.006	0.025	0.007	0.031
$6_\gamma^+ \to 6_g^+$	0.171	0.023	0.023	0.100	0.029	0.138

TABLE VIII: The same as in Table III but for ^{192}Os. The units for the B(E2) values are e^2b^2.

here has not only the merit of removing two drawbacks of the previous X(5) model (the functions in γ are not periodic) but also provides a better quantitative description.

V. CONCLUSIONS

Here we shall summarize the main results presented in the present paper. Starting with the differential equation for β and γ variables, involving also the Euler angles, provided by the liquid drop model, one may define analytically solvable equations for both deformation variables. This is possible under certain circumstances which are described in details in Sections II and III. The first model which achieved this situation corresponds to the so called $X(5)$ symmetry and has the merit of describing in a very simple fashion the properties of the critical point in shape phase transition. Two specific features are considered as drawbacks of the model:1)the function describing the variable γ are non-periodic functions and moreover are normalized to unity on a non-bounded interval. These two properties conflict the symmetry properties required by the starting liquid drop Hamiltonian. 2)In particular if we preserve the integration measure for γ to be $|\sin 3\gamma| d\gamma$ then the Hamiltonian depending on γ is not hermitian.

The scope of this paper was to remove these two drawbacks. The solution offered here is to describe the γ variable with spheroidal functions which are periodic in the interval $[0, 2\pi]$, with respect to the integration measure $|\sin 3\gamma| d\gamma$. Another solution would be the use of the Mathiew functions, but applications based on this solution are postponed for another publication. Both solutions lead the X(5) model in the limit of $|\gamma|$ small. As regards the β variable the description is performed with generalized Laguerre functions which are solutions of a Schrödinger equations involving the Davidson's potential. A particular case of this description is the situation when the centrifugal terms is vanishing, i.e. $\beta_0 = 0$. Of course that is the oscillator potential in β.

It is inferred that each of the solvable models in the β variable, which have been previously used as $E(5)$ models, may be associated to the spheroidal function description by separating the term proportional in $1/\beta^2$ and not depending on γ to renormalize the equation in β. Two examples are given in this paper where the Davidson potential and an infinite square well potential are used. Associations of other β potentials and the spheroidal function approach are presently under our consideration. The results will be published elsewhere.

The numerical applications to ^{150}Nd, ^{154}Gd and ^{152}Os reveal a good agreement with exerimental data. Moreover, the comparison with the results yielded by X(5) calculations suggest that the present approch provides a better quantitative description of the data. In the case of ^{154}Gd we compared the theoretical results of this paper with those obtained with two versions of the Coherent State Model. The qualities of the agreement with the experimental data obtained with the three sets of calculations are comparable with each other. The essential diference is that CSM works quite well for all even Gd isotopes while the $X(5)$-type models work only for the critical point of the shape phase transition.

Perhaps this is the prize we have to pay by endorsing the variable separability. The model beauty cannot substitute the virtue of the variable mixing to account for details specific for one phase or another.

Acknowledgments. This work was partly supported by CNCSIS/Romania under the contract ID33/2007.

VI. APPENDIX A

In order to help the reader to check the expressions given in the text, here we give some intermediate results concerning the partial expansions in γ, used in deriving the fourth order expansion for the Hamiltonian considered in Section II. Thus, the useful expansions are listed below:

$$\begin{aligned}
\frac{1}{\sin^2(\gamma-\frac{2\pi}{3})} &= \frac{4}{3}-\frac{8\gamma}{3\sqrt{3}}+\frac{8\gamma^2}{3}-\frac{16\gamma^3}{3\sqrt{3}}+\frac{104\gamma^4}{27}+\mathcal{O}[\gamma]^5,\\
\frac{1}{\sin^2(\gamma+\frac{4\pi}{3})} &= \frac{4}{3}+\frac{8\gamma}{3\sqrt{3}}+\frac{8\gamma^2}{3}+\frac{16\gamma^3}{3\sqrt{3}}+\frac{104\gamma^4}{27}+\mathcal{O}[\gamma]^5,\\
\frac{1}{\sin^2\gamma} &= \frac{1}{\gamma^2}+\frac{1}{3}+\frac{\gamma^2}{15}+\frac{2\gamma^4}{189}+\mathcal{O}[\gamma]^5,\\
\frac{1}{\sin^2(3\gamma)} &= \frac{1}{9\gamma^2}+\frac{1}{3}+\frac{3\gamma^2}{5}+\frac{6\gamma^4}{7}+\mathcal{O}[\gamma]^5,\\
\cos(3\gamma) &= 1-\frac{9\gamma^2}{2}+\frac{27\gamma^4}{8}+\mathcal{O}[\gamma]^5,\\
\cos^2(3\gamma) &= 1-9\gamma^2+27\gamma^4+\mathcal{O}[\gamma]^5.
\end{aligned} \tag{3.1}$$

In order to perform the expansion around $\pi/6$ one needs the following expansions in terms of $y=|\gamma-\pi/6|$:

$$\begin{aligned}
\frac{1}{\sin^2(\gamma-\frac{2\pi}{3})} &= 1+y^2+\frac{2y^4}{3}+\mathcal{O}[y]^5,\\
\frac{1}{\sin^2(\gamma+\frac{4\pi}{3})} &= 4+8\sqrt{3}y+40y^2+\frac{176y^3}{\sqrt{3}}+\frac{728y^4}{3}+\mathcal{O}[y]^5,\\
\frac{1}{\sin^2\gamma} &= 4-8\sqrt{3}y+40y^2-\frac{176y^3}{\sqrt{3}}+\frac{728y^4}{3}+\mathcal{O}[y]^5,\\
\frac{1}{\sin^2(3\gamma)} &= 1+9y^2+54y^4+\mathcal{O}[y]^5,\\
\cos(3\gamma) &= -3y+\frac{9y^3}{2}+\mathcal{O}[y]^5,\\
\cos^2(3\gamma) &= 9y^2-27y^4+\mathcal{O}[y]^5.
\end{aligned} \tag{3.2}$$

VII. APPENDIX B

In the rotational term:

$$W(\gamma,Q)=\frac{1}{4}\sum_{k=1}^{3}\frac{1}{\sin^2\left(\gamma-\frac{2\pi}{3}k\right)}Q_k^2, \tag{B.1}$$

we shall consider the identity:

$$\frac{1}{4}\sum_{k=1}^{3}\frac{1}{\sin^2\left(\gamma-\frac{2\pi}{3}k\right)}=\frac{9}{\sin^2(3\gamma)}. \tag{B.2}$$

The term $W(\gamma, Q)$ acquires a more convenient form:

$$\begin{aligned} W(\gamma, Q) &= \frac{1}{8}\left[\frac{9}{\sin^2(3\gamma)}-\frac{1}{\sin^2\gamma}\right](Q_1^2+Q_2^2+Q_3^2)+\frac{3}{8}\left[-\frac{3}{\sin^2(3\gamma)}+\frac{1}{\sin^2\gamma}\right]Q_3^2 \\ &+ \frac{1}{8}\left[\frac{1}{\sin^2(\gamma-\frac{1\pi}{3})}-\frac{1}{\sin^2(\gamma-\frac{4\pi}{3})}\right](Q_1^2-Q_2^2)\ . \end{aligned} \tag{B.3}$$

In the regime of $|\gamma| \ll 1$ one uses the expansions

$$\begin{aligned} \frac{9}{4\sin^2 3\gamma} &= \frac{1}{4\gamma^2}+\frac{3}{4}+\frac{27\gamma^2}{20}+\mathcal{O}(\gamma^3), \\ \frac{1}{\sin^2\gamma} &= \frac{1}{\gamma^2}+\frac{1}{3}+\frac{\gamma^2}{15}+\mathcal{O}(\gamma^3), \\ \cos 3\gamma &= 1-\frac{9\gamma^2}{2}+\mathcal{O}(\gamma^3), \\ \cos^2 3\gamma &= 1-9\gamma^2+\mathcal{O}(\gamma^3), \end{aligned} \tag{B.4}$$

in connection with the expression B.3 of $W(\gamma, Q)$. The result is:

$$\begin{aligned} W(\gamma, Q) &= \left(\frac{1}{3}+\frac{2}{3}\gamma^2\right)(Q_1^2+Q_2^2+Q_3^2)+\left(\frac{1}{4\sin^2\gamma}-\frac{1}{3}-\frac{2}{3}\gamma^2\right)Q_3^2 \\ &+ \frac{2}{3\sqrt{3}}(Q_2^2-Q_1^2)+\mathcal{O}(\gamma^3). \end{aligned} \tag{B.5}$$

Inserting this expression into the Hamiltonian

$$\tilde{H}-E=\frac{\partial^2}{\partial\gamma^2}+\frac{9}{4}\left[1+\frac{1}{\sin^2 3\gamma}\right]-U(\gamma)-W(\gamma, Q), \tag{B.6}$$

and averaging the result with an axially symmetric rotational state, for which $\langle Q_1\rangle = \langle Q_2\rangle$, and denoting the result by H_0, one obtains:

$$H_0=\frac{\partial^2}{\partial\gamma^2}+\frac{9}{4}-U(\gamma)-V(\gamma), \tag{B.7}$$

$$V(\gamma)=\frac{9}{8\sin^2(3\gamma)}\left[L(L+1)-q_3^2-2\right]-\frac{1}{8\sin^2\gamma}\left[L(L+1)-3q_3^2\right]. \tag{B.8}$$

Here we used the notation $q_k = \langle Q_k\rangle$. If the averaging is performed with a Wigner function D^L_{MK}, the result for $V(\gamma)$ would be:

$$V(\gamma)=\frac{1}{4\sin^2\gamma}(K^2-1)+\frac{1}{8}\left[\frac{9}{\sin^2(3\gamma)}-\frac{1}{\sin^2\gamma}\right]\left[L(L+1)-2-K^2\right]. \tag{B.9}$$

In the limit of $|\gamma| \ll 1$ the above expression of $V(\gamma)$ can be expanded in powers of γ. The second order expansion is:

$$V(\gamma) = \frac{1}{12}\left(\frac{1}{3\gamma^2} + 1 + \frac{1}{5}\gamma^2\right)(K^2-1) + \frac{1}{3}\left[L(L+1) - K^2 - 2\right](1+2\gamma^2) + \mathcal{O}(\gamma^3). \quad \text{(B.10)}$$

On the other hand making use of the expansion

$$\frac{9}{\sin^2 3\gamma} = \frac{1}{\sin^2\gamma} + \frac{8}{3}(1 + 2\sin^2\gamma) + \mathcal{O}(\gamma^3), \quad \text{(B.11)}$$

one obtains the following second order expansion in $\sin\gamma$:

$$V(\gamma) = \frac{K^2-1}{4\sin^2\gamma} + \frac{1}{3}\left[L(L+1) - K^2 - 2\right](1 + 2\sin^2\gamma) + \mathcal{O}(\gamma^3). \quad \text{(B.12)}$$

From this expression one obtains immediately the expansion in $\sin(3\gamma)$

$$V(\gamma) = \frac{9(K^2-1)}{4\sin^2 3\gamma} + \frac{1}{3}\left[L(L+1) - K^2 - 2\right]\left(1 + \frac{2}{9}\sin^2 3\gamma\right) + \mathcal{O}(\gamma^3). \quad \text{(B.13)}$$

The three expansions for $V(\gamma)$ are useful to study different representations for the wave function in γ. Thus, inserting (B.10) in Eq.(B.7) one obtains a differential equation for the Laguerre functions. Here, all corrections in γ^2 were included. Using the expansion in $\sin\gamma$ the differential equation for the variable γ becomes an equation for the spheroidal function after the change of variable $x = cos\gamma$ is performed. Finally, we mention that the use of Eq.(B.13) leads to a differential equation for the spheroidal functions in the variable $x = \cos(3\gamma)$.

VIII. APPENDIX C

The renormalization of the equation in β due to the terms coming form the rotational term, is based on the expansion in $\sin 3\gamma$ given by Eq. (B.13). Multiplying $V(\gamma)$ given by the quoted equation with $\frac{1}{\beta^2}$ one obtains terms which depend on γ and terms which do not depend on this variable. The latter terms are:

$$R = \frac{1}{3\beta^2}\left[L(L+1) - K^2 - 2\right]. \quad \text{(C.1)}$$

Our option for renormalization is based on the approximation:

$$R = \frac{1}{3\beta^2}L(L+1) - \frac{1}{3\langle\beta^2\rangle}\left[K^2 + 2\right] \quad \text{(C.2)}$$

In this way the indices for the generalized Laguerre functions, if we use the Davidson potential, or of Bessel function if we use an infinite square well γ potential are those used in the present paper and Ref.[49] If the separation of the two types of terms is achieved differently:

$$R = \frac{1}{3\beta^2}\left[L(L+1)-2\right] - \frac{1}{3\langle\beta^2\rangle}K^2 \tag{C.3}$$

then the irrational indices for Laguerre and Bessel functions are:

$$\begin{aligned} p &= -\frac{3}{2}+\sqrt{\frac{1}{3}\left(L+\frac{1}{2}\right)^2+\frac{3}{2}+\beta_0^4}, \\ s &= -\frac{3}{2}+\sqrt{\frac{1}{3}\left(L+\frac{1}{2}\right)^2+\frac{3}{2}}. \end{aligned} \tag{C.4}$$

Note that in the case the β potential is a harmonic oscillator and the γ potential is ignored, the separation of the β variable takes place in a natural manner, the γ and Euler angles depending terms being just the Casimir operator of the group $R(5)$. Due to this separation the β wave function is not depending on the quantum number K. Here such a simple picture does not hold any longer and a K depending term may renormalize the equation for β. Therefore, it becomes meaningful to consider the full term R (C.1) for the renormalization purpose. In this case the two irrational indices for the generalized Laguerre function and the Bessel function are:

$$\begin{aligned} p &= -\frac{3}{2}+\sqrt{\frac{1}{3}\left(L+\frac{1}{2}\right)^2-\frac{K^2}{3}+\frac{3}{2}+\beta_0^4}, \\ s &= -\frac{3}{2}+\sqrt{\frac{1}{3}\left(L+\frac{1}{2}\right)^2-\frac{K^2}{3}+\frac{3}{2}}. \end{aligned} \tag{C.5}$$

The difference between the index s given by Eq.(C.5) and that used in Ref.[47] is caused by the fact the therein the term $\frac{2}{3\beta^2}$ from Eq.(C.1) is not used for renormalizing the equation for β.

---

[1] A. Bohr, Mat. Fys. Medd. Dan. Vid. Selsk. **26** (1952) no.14; A.Bohr and B.Mottelson, Mat. Fys. Medd. Dan. Vid. Selsk. **27** (1953) no. 16

[2] A. Faessler and W. Greiner, Z. Phys. **168** (1962) 425; **170** (1962) 105; **177** (1964) 190; A. Faessler, W. Greiner and R. Sheline, Nucl. Phys. **70** (1965) 33.

[3] G. Gneus, U. Mosel and W. Greiner, Phys. Lett. **30 B** (1969) 397.

[4] P. Hess, J. Maruhn and W. Greiner, Phys. Rev. **C23** (1981) 2335; J. Phys. G, **7** (1981) 737.

[5] A. A. Raduta, V. Ceausescu, A. Gheorghe and R. M. Dreizler, Phys. Lett. **99B** (1981) 444; Nucll. Phys. **A381** (1982) 253.

[6] A. A. Raduta, V. Ceausescu and A. Faessler, Phys. Rev. **C36** (1987) 2111.

[7] A. A. Raduta, C. Lima and A. Faessler, Z. Phys. **A 313** (1983) 69.

[8] A. A. Raduta, Al. H. Raduta and A. Faessler, Phys. Rev. **C55** (1997) 1747; A. A. Raduta, D. Ionescu and A. Faessler, Phys. Rev. **C65** (2002) 064322.

[9] A. A. Raduta, in Recent Res. Devel. Nuclear Phys.,1 (2004):1-70, ISBN:81-7895-124-X.

[10] L. Wilets and M. Jean, Phys. Rev. **102** (1956) 788.

[11] A. S. Davydov and G. F. Filippov, Nucl. Phys. **8** (1958) 788.

[12] A. Arima and F. Iachello, Ann. Phys.(N.Y.) **99** (1976) 253; **123** (1979) 468.

[13] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, England, 1987).

[14] R. F. Casten, in Interacting Bose-Fermi Systems in Nuclei, edited by F. Iachello (Plenum, New York, 1981), p. 1.

[15] J. H. Ginocchio and M. W. Kirson, Phys. Rev. Lett.**44**(1980) 1744.

[16] A. E. L. Dieperink, O. Scholten snd F. Iachello, Phys. Rev. Lett.**44** (1980) 1767.

[17] F. Iachello, Phys. Rev. Lett. **85** (2000) 3580.

[18] F. Iachello, Phys. Rev. Lett. **87** (2001) 052502.

[19] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85** (2000) 3584.

[20] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87** (2001) 052503.

[21] R. M. Clark, M. Cromaz, M. A. Deleplanque, M. Descovich, R. M. Diamond, P. Fallon, I. Y. Lee, A. O. Macchiavelli, H. Mahmud, E. Rodriguez-Vieitez, F. S. Stephens and D. Ward, Phys. Rev. **C 69** (2004) 064322.

[22] N. V. Zamfir et al., Phys. Rev. **C65** (2002) 044325.

[23] Da-li Zhang and Yu-xin Liu, Phys. Rev. **C65** (2002) 057301.

[24] Lorenzo Fortunato and Andrea Vitturi, Jour. Phys. G:Nucl. Part. Phys. **29** (2003) 1341.

[25] Dennis Bonatsos, D. Lenis, N. Minkov, D. Petrellis, P. P. Raychev, P. A. Terziev, arXiv: nucl-th/0312121 v1, 29 Dec 2003, Phys. Lett. $\underline{5}$84 (2004) 40.

[26] P. M. Davidson, Proc. R. Soc. **135** (1932) 459.

[27] A. A. Raduta, A. Gheorghe and A. Faessler, J. Phys. G:Nucl. Part. Phys.,**31** (2005) 337.

[28] L. Fortunato, arXiv:Nucl-Th/0411087.

[29] A. Gheorghe, A. A. Raduta and V. Ceausescu, Nucl. Phys. **A 296** (1978) 228.

[30] A. A. Raduta, A. Gheorghe and V. Ceausescu, Nucl. Phys. **A311** (1978) 118.

[31] A. Gheorghe, A. A. Raduta and V. Ceausescu, Nucl. Phys. **A637** (1998) 201.

[32] A. Gheorghe, A. A. Raduta and A. Faessler, Phys. Lett. **B 648**, 171, (2007).

[33] P. M. Davidson, Proc. R. Soc. **135** (1932) 459.

[34] J. P. Elliot, J. A. Evans, P. Park, Phys. Lett. **B 169** (1986) 309.

[35] D. J. Rowe, C. Bahri, J. Phys. **A 31** (1998) 4947

[36] M. Abramowitz, and I. A. Stegun, (Eds.), Handbook of Mathematical Functions with Formulas,Graphs, and Mathematical Tables, 9th printing. New York: Dover,1972, pp. 751-759.

[37] R. Krücken *et al* Phys. Rev. Lett. **88** (2002) 232501-1.

[38] E. der Mateosian, J. K. Tuli, NDS **75** (1995) 827.

[39] R. M. Clark **et al**, Phys. Rev. **C68** (2003) 037301.

[40] P. O. Lipas *et al* Phys. Scr. 7 (1983) 8.

[41] C. Girit, W. D. Hamilton and C. A. Katfas, J. Phys. G: Nucl. Part. Phys.**9** (1983) 797.

[42] C. W. Reich, R. G. Helmer, NDS **85** (1998) 171.

[43] A. A. Raduta and Amand Faessler, J. Phys. G: Nucl. Part. Phys. **31** (2005) 873.

[44] J. M. Allmond *et al.* Phys. Rev. **C 78** (2008) 014302.

[45] Coral M. Baglin, NDS **84** (1998) 717.

[46] G. A. Lalazissis, S. Raman and P. Ring, At. Data Nucl. Data Tables, **71** (1999) 1.

[47] R. Bijker, R. F. Casten, N. V. Zamfir, E. A. McCutchan, Phys. Rev. **C 68** (2003) 064304.

[48] M. A. Caprio, Phys. Rev. **C69** (2004) 044307.

[49] D. Bonatsos, D. Lenis, E. A. McCutchan, D. Petrellis, I. Yigitoglu, Phys. Lett. **B649** (2007) 394.

[50] S.G.Nilsson,Mat.Fys.Medd. K. Dan. Vid.Selsk.**29** no. 16 (1955).

[51] A. A. Raduta, D. S. Delion and N. Lo Iudice, Nucl. Phys. **A 564** (1993) 185.

[52] G. Palma and U. Raff Amm. J. Phys. **71** (2003) 956.