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# Towards a new solvable model for the even-even triaxial nuclei

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**Abstract.** The even-even triaxial nuclei are described by amending the Bohr-Mottelson Hamiltonian with an energy potential consisting of two terms: a sextic oscillator with centrifugal barrier in the  $\beta$  variable and a periodic function in the  $\gamma$  variable. After the variable separation is performed, the  $\beta$  equation is quasi-exactly solved, while the  $\gamma$  equation is satisfied by the Mathieu function. The reduced E2 transition probabilities are determined using an anharmonic transition operator. The formalism is conventionally called the Sextic and Mathieu Approach (SMA). Numerical applications concerned seven non-axial nuclei:  $^{188}\text{Os}$ ,  $^{190}\text{Os}$ ,  $^{192}\text{Os}$ ,  $^{228}\text{Th}$ ,  $^{230}\text{Th}$ ,  $^{182}\text{W}$  and  $^{180}\text{Hf}$ . SMA results are compared with the experimental data as well as with those yielded by the Coherent State Model (CSM).

## 1. Introduction

The field of the nuclear shape phase transitions received a considerable attention when it was noticed that the critical points may be described by differential equations which are exactly solvable. In some of the situations the solutions of these equations achieve the irreducible representations of a certain symmetry group. For example the E(5) symmetry [1] describes the critical point for the transition U(5)→O(6) while the one associated to the transition U(5)→SU(3) is not yet known and thereby referred to as X(5) [2]. Since the proposal for the two critical points showed up, on that matter a huge number of papers were published, many of them being reviewed in Refs. [3, 4]. In the mean time other two critical symmetries were proposed, namely Y(5) [5] and Z(5) [6], for the axial-triaxial shape phase transition and for the prolate-oblate shape phase transition, respectively. All these critical point symmetries are analytical solutions of the Bohr-Mottelson Hamiltonian [7, 8] amended with a potential which depends on both  $\beta$  and  $\gamma$  variables.

Here, we present a new solution of the Bohr-Mottelson Hamiltonian equation which seems to be suitable for the description of the even-even triaxial nuclei having an axial deformation close to  $\gamma_0 = 30^\circ$ . In order, to improve the agreement between the theoretical predictions and the corresponding experimental data, we used for the  $\beta$  variable a sextic oscillator with centrifugal barrier potential. The  $\beta$  equation with sextic potential is quasi-exactly solvable, which means that is still exactly solvable but for a finite number of states. For the  $\gamma$  equation, choosing a periodic potential with a minimum in  $\gamma_0 = 30^\circ$  and avoiding the approximations made in the previous models, we obtained as solutions the Mathieu functions which are periodic in the interval  $[0, 2\pi]$ . In this way, the hermiticity of the  $\gamma$  Hamiltonian in respect with the integration

measure  $|\sin 3\gamma|d\gamma$  is preserved. The reduced E2 transition probabilities were determined using a transition operator written in the intrinsic reference frame and which contains two terms, a harmonic part and an anharmonic part, respectively. The formalism developed in this way was conventionally called the Sextic and Mathieu Approach (SMA). More details about the SMA and its numerical applications may be found in the Refs. [9, 10, 11]. Here, we present only its main ingredients and some numerical examples. The SMA results were also compared with those yielded by the Coherent State Model (CSM) [12]. Numerical analysis of the two models results suggested a possible relationship between the two approaches. The connection between SMA and CSM was analytically established in Refs. [10, 11], where the SMA equations were obtained through a semi-classical treatment of the CSM Hamiltonian.

The description of the results presented in this communication is organized as follows. In Section II, the main ingredients of the SMA model are given, while in Section III an illustrative example of  $^{192}\text{Os}$  is discussed. In Section IV we show how the SMA equations were obtained from the CSM Hamiltonian and finally, in Section IV, the main conclusions are summarized.

## 2. The Sextic and Mathieu Approach

In order to describe the critical nuclei of the prolate-oblate shape phase transition, we invoke the Bohr-Mottelson Hamiltonian with a potential depending on both the  $\beta$  and  $\gamma$  variables:

$$H\psi(\beta, \gamma, \Omega) = E\psi(\beta, \gamma, \Omega), \text{ where,} \quad (1)$$

$$H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1}^3 \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] + V(\beta, \gamma),$$

$$V(\beta, \gamma) = V_1(\beta) + \frac{V_2(\gamma)}{\beta^2}. \quad (2)$$

Here,  $\beta$ ,  $\gamma$  and  $\Omega$  are the intrinsic deformation variables and the Euler angles, respectively, while with  $Q_k$  are denoted the angular momentum projections in the intrinsic reference frame. After the separation of variables [9], the following equations are obtained:

$$\left[ -\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{L(L+1)}{\beta^2} + v_1(\beta) \right] f(\beta) = \varepsilon_\beta f(\beta), \quad (3)$$

$$\left[ -\frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{3}{4}R^2 + \left( 10L(L+1) - \frac{39}{4}R^2 \right) \left( \gamma - \frac{\pi}{6} \right)^2 + v_2(\gamma) \right] \phi(\gamma) = \tilde{\varepsilon}_\gamma \phi(\gamma), \quad (4)$$

where the following notations are used:

$$v_1(\beta) = \frac{2B}{\hbar^2} V_1(\beta), \quad v_2(\gamma) = \frac{2B}{\hbar^2} V_2(\gamma), \quad \varepsilon_\beta = \frac{2B}{\hbar^2} E_\beta, \quad \tilde{\varepsilon}_\gamma = \langle \beta^2 \rangle \frac{2B}{\hbar^2} E_\gamma. \quad (5)$$

Eq. (3) is reduced to the sextic oscillator equation if we change the function  $f(\beta) = \beta^{-2}\varphi(\beta)$  and take the  $\beta$  potential in the form:

$$v_1^\pi(\beta) = (b^2 - 4ac^\pm)\beta^2 + 2ab\beta^4 + a^2\beta^6 + u_0^\pm c^\pm = \frac{L}{2} + \frac{5}{4} + M, \quad M = 0, 1, 2, \dots, \quad (6)$$

where,  $c^\pm$  and  $u_0^\pm$  are constants with the signs + and - for  $L$  even and  $L$  odd, respectively. The solutions of Eq. (3), with the above specified potential are

$$\varphi_{n_\beta, L}^{(M)}(\beta) = N_{n_\beta, L} P_{n_\beta, L}^{(M)}(\beta^2) \beta^{L+1} e^{-\frac{a}{4}\beta^4 - \frac{b}{2}\beta^2}, \quad n_\beta = 0, 1, 2, \dots, M, \quad (7)$$

where  $N_{n_\beta, L}$  are the normalization factor, while  $P_{n_\beta, L}^{(M)}(\beta^2)$  are polynomials in  $x^2$  of  $n_\beta$  order. The corresponding excitation energies are:

$$E_\beta(n_\beta, L) = \frac{\hbar^2}{2B} \left[ b(2L + 3) + \lambda_{n_\beta}^{(M)}(L) + u_0^\pm \right], \quad n_\beta = 0, 1, 2, \dots, M, \quad (8)$$

where  $\lambda_{n_\beta}^{(M)} = \varepsilon_\beta - u_0^\pm - 4bs$  is the eigenvalue of the equation:

$$\left[ - \left( \frac{\partial^2}{\partial \beta^2} + \frac{4s-1}{\beta} \frac{\partial}{\partial \beta} \right) + 2b\beta \frac{\partial}{\partial \beta} + 2a\beta^2 \left( \beta \frac{\partial}{\partial \beta} - 2M \right) \right] P_{n_\beta, L}^{(M)}(\beta^2) = \lambda_{n_\beta}^{(M)} P_{n_\beta, L}^{(M)}(\beta^2). \quad (9)$$

Considering the periodic  $\gamma$  potential,  $v_2(\gamma) = \mu \cos^2 3\gamma$  (4), which exhibits a minimum in  $\gamma_0 = \pi/6$  and then changing the function  $\phi(\gamma) = \mathcal{M}(3\gamma)/\sqrt{|\sin 3\gamma|}$  one arrives at a differential equation for the Mathieu functions. The expression for the excitation energy of the  $\gamma$  equation is:

$$E_\gamma(n_\gamma, L, R) = \frac{\hbar^2}{2B} \frac{1}{\langle \beta^2 \rangle} \left[ 9a_{n_\gamma}(L, R) + 18q(L, R) - \frac{3}{4}R^2 - \frac{5}{2} \right], \quad n_\gamma = 0, 1, 2, \dots \quad (10)$$

The Mathieu functions have the advantage that are periodic and defined on a bounded interval  $[0, 2\pi]$ , preserving the hermiticity of the initial  $\gamma$  Hamiltonian in respect with the integration measure  $|\sin 3\gamma|d\gamma$ . The total energy of the system is obtain by adding the contributions coming from both  $\beta$  and  $\gamma$  equations.

Using the Rose's convention [15], the reduced E2 transition probabilities are determined by:

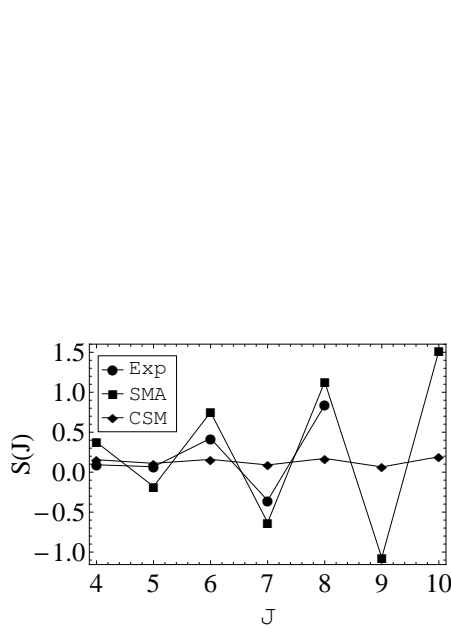
$$B(E2, L_i \rightarrow L_f) = |\langle L_i || T_2^{(E2)} || L_f \rangle|^2, \quad (11)$$

where the transition operator has the following expression:

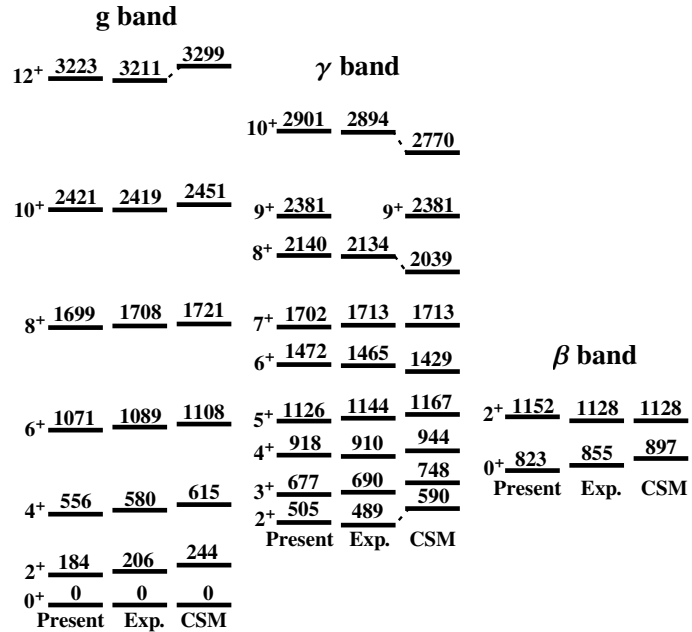
$$T_{2\mu}^{(E2)} = t_1\beta \left[ \cos \left( \gamma - \frac{2\pi}{3} \right) D_{\mu 0}^2 + \frac{1}{\sqrt{2}} \sin \left( \gamma - \frac{2\pi}{3} \right) (D_{\mu 2}^2 + D_{\mu, -2}^2) \right] + t_2\sqrt{\frac{2}{7}}\beta^2 \left[ -\cos \left( 2\gamma - \frac{4\pi}{3} \right) D_{\mu 0}^2 + \frac{1}{\sqrt{2}} \sin \left( 2\gamma - \frac{4\pi}{3} \right) (D_{\mu 2}^2 + D_{\mu, -2}^2) \right]. \quad (12)$$

### 3. Numerical results

SMA was successfully applied for several non-axial nuclei:  $^{188}\text{Os}$ ,  $^{190}\text{Os}$ ,  $^{192}\text{Os}$ ,  $^{228}\text{Th}$ ,  $^{230}\text{Th}$ ,  $^{182}\text{W}$  and  $^{180}\text{Hf}$  [9, 10, 11]. For the sake of saving the space here we shall present only the case of  $^{192}\text{Os}$ . This nucleus is a good candidate for a triaxial deformation close to  $\gamma_0 = 30^\circ$ . Indeed, its equilibrium value predicted by Leander [16] is  $\gamma_0 = 25^\circ$ . On the other hand one signature of the triaxial rigid rotor is  $\Delta E = |E_{2_1^+} + E_{2_2^+} - E_{3_1^+}| = 0$ . In the chosen case this equality is obeyed with a good accuracy having  $\Delta E = 5\text{keV}$ . Moreover, its  $\gamma$  band has a pronounced staggering behavior (Fig.1). As seen in Fig. 1 and Fig. 2, the staggering and the spectrum of  $^{192}\text{Os}$  are quite well described by SMA. The agreement with the data is appraised by the value of the *r.m.s.* value associated to the predicted energy deviation, which in the mentioned case amounts of 16 keV. The calculated E2 properties regard the intraband transitions of the ground and  $\gamma$  bands, as well as the interband  $\gamma \rightarrow g$ . Results are compared with the corresponding experimental data in Table 1. Comparing the SMA results with those of CSM, we noticed that they are quantitatively close to each other, although the two approaches have apparently different grounds. The connection between SMA and CSM was explained in details in Refs. [10, 11].



**Figure 1.** The staggering behavior of  $^{192}\text{Os}$  (Exp) compared with the SMA and CSM predictions.



**Figure 2.** The experimental spectrum of  $^{192}\text{Os}$  [17, 18] for the first three bands, given in keV units, is compared with the results yielded by the SMA (Present) and CSM.

**Table 1.** Some  $B(E2)$  values for  $^{192}\text{Os}$  obtained with SMA and CSM, are compared with the corresponding experimental data [17].

$B(E2)[e^2b^2]$	Exp	SMA	CSM
$2^+_g \rightarrow 0^+_g$	0.424	0.424	0.236
$4^+_g \rightarrow 2^+_g$	0.497	0.632	0.449
$6^+_g \rightarrow 4^+_g$	0.660	0.858	0.611
$8^+_g \rightarrow 6^+_g$	0.754	1.030	0.754
$10^+_g \rightarrow 8^+_g$	0.688	1.175	0.887
$4^+_γ \rightarrow 2^+_γ$	0.298	0.261	0.277
$6^+_γ \rightarrow 4^+_γ$	0.336	0.352	0.595
$8^+_γ \rightarrow 6^+_γ$	0.314	0.549	0.814
$2^+_γ \rightarrow 0^+_g$	0.037	0.006	0.192
$2^+_γ \rightarrow 2^+_g$	0.303	0.303	0.055
$2^+_γ \rightarrow 4^+_g$	0.024	0.000	0.000
$4^+_γ \rightarrow 2^+_g$	0.002	0.004	0.274
$4^+_γ \rightarrow 4^+_g$	0.203	0.068	0.137
$4^+_γ \rightarrow 6^+_g$	0.018	0.000	0.000
$6^+_γ \rightarrow 4^+_g$	0.000	0.002	0.357
$6^+_γ \rightarrow 6^+_g$	0.171	0.042	0.171

#### 4. The connection between the SMA and CSM formalisms

Aiming at obtaining the SMA equations starting with CSM, first the CSM boson Hamiltonian is dequantized using a two parameter coherent state function as a trial variational function:

$$\mathcal{H} = \langle \psi | H_{\text{CSM}} | \psi \rangle, \quad |\psi\rangle = \exp \left[ z_0 b_0^\dagger + z_2 b_2^\dagger + z_{-2} b_{-2}^\dagger - z_0^* b_0 - z_2^* b_2 - z_{-2}^* b_{-2} \right] |0\rangle, \quad (13)$$

The basic property of the coherent state is:

$$b_{\mu}|\psi\rangle = (\delta_{\mu,0}z_0 + \delta_{\mu,2}z_2 + \delta_{\mu,-2}z_{-2})|\psi\rangle. \quad (14)$$

New coordinates which bring the classical equations to the Hamilton canonical form are desirable. Such coordinates are obtained by the restriction  $z_2 = z_{-2}$  and the transformation  $q_0 = \sqrt{2}\text{Re}[z_0]$ ,  $p_0 = \sqrt{2}\text{Im}[z_0]$ ,  $q_2 = 2\text{Re}[z_2]$ ,  $p_2 = 2\text{Im}[z_2]$ . In the expression of the classical Hamilton energy function, the terms which couple the coordinate with momentum and also those which are not quadratic in momenta are neglected. The resulting Hamiltonian is quantized in polar coordinates which results in obtaining:

$$\hat{H} = - \left( 11A_1 + 3A_2 + A_1d^2 + \frac{3}{70}d^4A_3 \right) \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \gamma^2} \right) + V_1(r) + V_2(\gamma), \quad (15)$$

$$V_1(r) = \left[ 11A_1 + 3A_2 - \frac{3d^2}{4}A_1 - \frac{3A_3}{70}d^4 \right] r^2 + \left[ \frac{A_1}{4} + \frac{9A_3}{280}d^2 \right] r^4 + \frac{A_3}{280}r^6, \quad V_2(\gamma) = \frac{A_3r_0^6}{280} \cos 6\gamma. \quad (16)$$

Finally, the SMA equations are obtained after the separation of variables  $r = k\beta$  and  $\gamma$  through Taylor expansions around the equilibrium values,  $\beta_0$  and  $\gamma_0$ .

## 5. Conclusions

A new formalism, called the Sextic and Mathieu Approach (SMA), aimed at describing the even-even triaxial nuclei was proposed. The  $\beta$  equation is that of a sextic oscillator with centrifugal barrier, while that of the  $\gamma$  variable is reduced to a Mathieu equation. The comparison between the SMA results and experimental data of seven nuclei  $^{188,190,192}\text{Os}$ ,  $^{228,230}\text{Th}$ ,  $^{180}\text{Hf}$  and  $^{182}\text{W}$  showed a good agreement. Also, a connection between the SMA and CSM formalism was pointed out. The closeness of the SMA and CSM results was explained by obtaining the SMA equations through a semi-classical treatment of the CSM Hamiltonian. With this demonstration, the SMA potentials get a theoretical support in contrast with their intuitive choice when the SMA equations emerge from the BohrMottelson Hamiltonian.

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