Application of the sextic oscillator potential together with Mathieu and spheroidal functions for triaxial and X(5) type nuclei

Apolodor A. Raduta and Petrică Buganu

Horia Hulubei National Institute of Physics and Nuclear Engineering, Bucharest-Magurele, RO-077125, Romania, EU

Introduction

The Bohr-Mottelson model [1] describes collective low-lying states of the quadrupole heavy nuclei in terms of vibrations and rotations of the nuclear surface:

$$R_k = R_0\left[1+\frac{5}{4\pi}\beta\cos\left(\gamma-\frac{2\pi}{3}k\right)\right],\quad k=1,2,3. \qquad (1)$$

Here, R_k are the radii of the ellipsoid, R_0 is the radius of the spherical nucleus, while β and γ denote the intrinsic deformation coordinates. For $\beta=0$, in Eq.(1), we obtain a sphere while for $\beta\neq 0$ the shape is an ellipsoid as in Fig.1 [2].

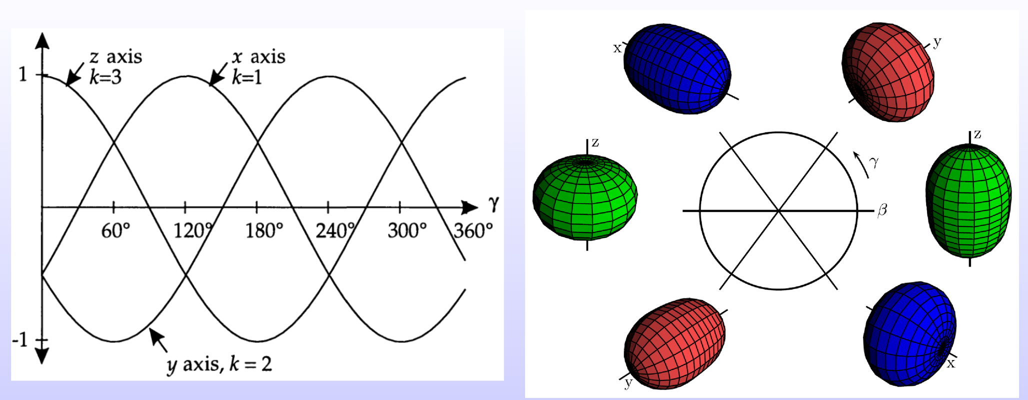


FIGURE 1: For $\gamma=0^0, 120^0, 240^0, 360^0$ and $\gamma=60^0, 180^0, 300^0$ we get prolate and oblate shapes, respectively. Between this γ values a triaxial shape appears.

The energy potential of the generalized Bohr-Mottelson Hamiltonian [3],

$$\hat{H} = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2}\left(\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\sum_{k=1}^{3}\frac{\hat{Q}_k^2}{4\sin^2(\gamma-\frac{2\pi}{3}k)}\right)\right] + V(\beta,\gamma), \qquad (2)$$

depends on both β and γ variables, in order to describe oscillations around deformed equilibrium shapes. Here, with Q_k are denoted the intrinsic angular momentum components. A great interest in solving the eigenvalue problem of the Hamiltonian given by Eq. (2) appeared when nuclei being close to the critical points of some shape phase transition were very well described by analytical solutions of it. The E(5) [4] solution describes the critical point of the transition between spherical and $\gamma-$unstable shape phase, while the one associated to the transition between spherical and symmetric shape phase is called X(5) [5]. Other two solutions for critical points were proposed short after that, namely Y(5) [6] and Z(5) [7], for the axial-triaxial shape phase transition and for the prolate-oblate shape phase transition, respectively.

In the present poster, we present new interesting solutions for the Hamiltonian (2), namely, Sextic and Mathieu Approach (SMA) [8,9,10] and Sextic and Spheroidal Approach (SSA) [11], respectively. SMA represents a realistic tool for the description of triaxial nuclei having axial deformations close to $\pi/6$, while SSA works very well for X(5) candidate nuclei.

New solutions for the generalized Bohr-Mottelson Hamiltonian

The separation of variables

The Bohr-Mottelson Hamiltonian [1] is amended with a potential which depends on both β and γ deformation variables [12,13],

$$V(\beta,\gamma) = V_1(\beta)+\frac{V_2(\gamma)}{\beta^2}, \qquad (3)$$

which allows us to separate the β variable from the γ variable and the three Euler angles θ_1,θ_2 and θ_3, which are still coupled due to the rotational term:

$$\hat{W} = \frac{1}{4}\sum_{k=1}^{3}\frac{\hat{Q}_k^2}{\sin^2\left(\gamma-\frac{2\pi}{3}k\right)}. \qquad (4)$$

Further, by performing a second order expansion of the rotational term W around $\gamma_0=0$ and $\gamma_0=\pi/6$ for X(5) type nuclei and triaxial nuclei respectively, and then averaging the resulting terms with specific Wigner functions, a complete separation of variables is achieved. The expansion is done such that the periodicity of the γ Hamiltonian to be preserved. The resulted equations are:

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{L(L+1)}{\beta^2}+v_1(\beta)\right]f(\beta)=\varepsilon_\beta f(\beta), \qquad (5)$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\tilde{W}+v_2(\gamma)\right]\phi(\gamma)=\tilde{\varepsilon}_\gamma\phi(\gamma), \qquad (6)$$

where the following notations are used:

$$v_1(\beta)=\frac{2B}{\hbar^2}V_1(\beta),\quad v_2(\gamma)=\frac{2B}{\hbar^2}V_2(\gamma),\quad \varepsilon_\beta=\frac{2B}{\hbar^2}E_\beta,\quad \tilde{\varepsilon}_\gamma=\langle\beta^2\rangle\frac{2B}{\hbar^2}E_\gamma. \qquad (7)$$

When the rotational term is expanded around $\gamma_0=0$ we have

$$\tilde{W}=2-\frac{2}{3}L(L+1)+\left(\frac{1}{4\sin^2\gamma}-\frac{1}{3}\right)K^2+\frac{2}{3}[L(L+1)-k^2]\gamma^2, \qquad (8)$$

while around $\gamma_0=\pi/6$ we have

$$\tilde{W}=2-\frac{3}{4}R^2+\left(10L(L+1)-\frac{39}{4}R^2\right)\left(\gamma-\frac{\pi}{6}\right)^2. \qquad (9)$$

$L(L+1)$, R and K are the eigenvalues of the total intrinsic angular momentul $\hat{Q}$ and of its projections on the axis 1 and 3, respectively.

Solution of the β equation

The Schrödinger equation for the β variable is quasi-exactly solved. Making the change of funtion $f(\beta)=\beta^{-2}\varphi(\beta)$ we have:

$$\left[-\frac{\partial^2}{\partial\beta^2}+\frac{L(L+1)}{\beta^2}+v_1(\beta)\right]\varphi(\beta)=\varepsilon_\beta\varphi(\beta). \qquad (10)$$

A sextic oscillator with centrifugal barrier potential is considered for the β equation, in order to realistically describe the experimental data of the well deformed nuclei:

$$v_1^{\pm}(\beta)=(b^2-4ac^{\pm})\beta^2+2ab\beta^4+a^2\beta^6+u_0^{\pm},\ c^{\pm}=\frac{L}{2}+\frac{5}{4}+M. \qquad (11)$$

Here, c is a constat which has two different values, one for L even and other for L odd:

$$(M,L):(k,0);(k-1,2);(k-2,4);...\Rightarrow c=k+\frac{5}{4}\equiv c^{+}\ (L\text{-even}), \qquad (12)$$

$$(M,L):(k,1);(k-1,3);(k-2,5);...\Rightarrow c=k+\frac{7}{4}\equiv c^{-}\ (L\text{-odd}). \qquad (13)$$

The constants $u_0^{\pm}$ are fixed such that the potential for L odd to have the same minimum energy with the potential for L even. The solutions of Eq. (10), with the potential given by the Eq. (11), are

$$\varphi^{(M)}_{n_\beta,L}(\beta)=N_{n_\beta,L}P^{(M)}_{n_\beta,L}(\beta^2)\beta^{L+1}e^{-\frac{a}{4}\beta^4-\frac{b}{2}\beta^2},\ n_\beta=0,1,2,...M, \qquad (14)$$

where $N_{n_\beta,L}$ are the normalization factor, while $P^{(M)}_{n_\beta,L}(\beta^2)$ are polynomials in x^2 of n_β order. The corresponding excitation energy is:

$$E_\beta(n_\beta,L)=\frac{\hbar^2}{2B}\left[b(2L+3)+\lambda^{(M)}_{n_\beta}(L)+u_0^{\pm}\right],\ n_\beta=0,1,2,...,M, \qquad (15)$$

where $\lambda^{(M)}_{n_\beta}=\varepsilon_\beta-u_0^{\pm}-4bs$ is the eigenvalue of the equation:

$$\left[-\frac{\partial^2}{\partial\beta^2}-\frac{4s-1}{\beta}\frac{\partial}{\partial\beta}+2b\beta\frac{\partial}{\partial\beta}+2a\beta^2\left(\beta\frac{\partial}{\partial\beta}-2M\right)\right]P^{(M)}_{n_\beta,L}=\lambda^{(M)}_{n_\beta}P^{(M)}_{n_\beta,L}. \qquad (16)$$

Solutions of the γ equations

Concerning the γ equation, its solution depends if we consider axial symmetric or triaxial nuclei. The potential in γ is chosen such that to exhibit minima in $\gamma=0$ and $\gamma=\pi/6$:

$$v_2(\gamma)=u_1\cos 3\gamma+u_2\cos^2 3\gamma. \qquad (17)$$

Performing a second order expansion around $\gamma_0=0$ in $\sin 3\gamma$ of $v_2(\gamma)$ and of the terms coming from the rotational term and then making the change of variable $x=\cos 3\gamma$ in Eq. (6) we obtain the spheroidal equation [14]:

$$\left[(1-x^2)\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}+\lambda_{m_\gamma,n_\gamma}-c^2x^2-\frac{m_\gamma^2}{1-x^2}\right]S_{m_\gamma,n_\gamma}(x)=0, \qquad (18)$$

$$\lambda_{m_\gamma,n_\gamma}=\frac{1}{9}\left[\tilde{\varepsilon}_\gamma-\frac{u_1}{2}-\frac{11}{27}D+\frac{1}{3}L(L+1)\right]+\frac{2L(L+1)}{27},$$
$$c^2=\frac{1}{9}\left(\frac{u_1}{2}+u_2-\frac{2}{27}D\right), \qquad (19)$$
$$m_\gamma=\frac{K}{2},\quad D=L(L+1)-K^2-2.$$

For triaxial nuclei, setting $u_1=0$ and expanding this time around $\pi/6$, after some steps, we get the Mathieu equation [8]:

$$\left(\frac{\partial^2}{\partial y^2}+a-2q\cos 2y\right)\mathcal{M}(y)=0,\quad y=3\gamma, \qquad (20)$$

$$q=\frac{1}{36}\left(\frac{10}{9}L(L+1)-\frac{13}{12}R^2+\mu-\frac{9}{4}\right),\ a=\frac{1}{9}\left(\tilde{\varepsilon}_\gamma+\frac{3}{4}R^2+\frac{5}{2}\right)-2q. \qquad (21)$$

The γ functions are normalized to unity with the integration measure $|\sin 3|\gamma d\gamma$ as the Bohr-Mottelson model requires:

$$\frac{3(2n_\gamma+1)(n_\gamma-m_\gamma)!}{2(n_\gamma+m_\gamma)!}\int_0^{\frac{\pi}{3}}|S_{m_\gamma,n_\gamma}(\cos 3\gamma)|^2|\sin 3\gamma|d\gamma=1. \qquad (22)$$

$$N^2_{L,R,n_\gamma}\int_0^{2\pi}|\phi_{L,R,n_\gamma}(\gamma)|^2|\sin 3\gamma|d\gamma=\frac{6}{\pi}\int_0^{\frac{\pi}{3}}|M_{L,R,n_\gamma}(3\gamma)|^2d\gamma=1. \qquad (23)$$

The total energy of the nuclear system is obtained by adding the contributions coming from the β and the γ equations.

Electromagnetic tranzitions

The reduced E2 transition probabilities are determined using the following formula:

$$B(E2;L_i\to L_f)=|\langle L_i||T_2^{(E2)}||L_f\rangle|^2, \qquad (24)$$

where
$$T^{(E2)}_{2\mu}=t_1\beta\left[\cos\gamma D^2_{\mu 0}+\frac{\sin\gamma}{\sqrt{2}}(D^2_{\mu 2}+D^2_{\mu,-2})\right]+t_2\sqrt{\frac{2}{7}}\beta^2\left[-\cos 2\gamma D^2_{\mu 0}+\frac{\sin 2\gamma}{\sqrt{2}}(D^2_{\mu 2}+D^2_{\mu,-2})\right]. \qquad (25)$$

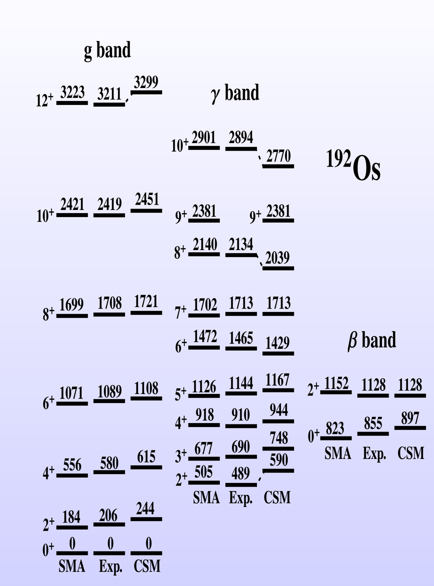
For triaxial nuclei, in the expression of the transition operetor (25) γ is substituted with $\gamma-2\pi/3$.

The models developed in this way are conventionally called the Sextic and Spheroidal Approach (SSA) and the Sextic and Mathieu Approach (SMA).

Numerical results

In Refs. [8,9,10], the SMA was successfully applied for several triaxial nuclei, ^{188}Os, ^{190}Os, ^{192}Os, ^{228}Th, ^{230}Th, ^{180}Hf and ^{182}W, chosen according to a certain signature of the rigid triaxial rotor. In Ref. [11], a good agreement of the SSA results with experimental data of several X(5) candidate nuclei as ^{176}Os, ^{178}Os, ^{180}Os, ^{188}Os, ^{190}Os, ^{150}Nd, ^{170}W, ^{156}Dy, ^{166}Hf and ^{168}Hf, was obtained. In Ref. [11], the SSA results were compared with those yielded by X(5), ISW [14], D [14] and Coherent State Model (CSM) [15]. From space reasons we present here only an example of nucleus for each of the models, SMA and SSA.

For SMA, the numerical results for ^{192}Os are shown in Fig. 2. Both, energetic spectrum and reduced probability transitions are very well explained by the SMA and CSM. Also, the staggering behavior of the γ band is reproduced by the SMA.



$B(E2; J_i^+\to J_f^+)\,[(eb)^2]$	Exp.	SMA	CSM
$2_g^+\to 0_g^+$	0.424	0.424	0.236
$4_g^+\to 2_g^+$	0.497	0.632	0.449
$6_g^+\to 4_g^+$	0.660	0.858	0.611
$8_g^+\to 6_g^+$	0.754	1.030	0.754
$10_g^+\to 8_g^+$	0.688	1.175	0.887
$4_\gamma^+\to 2_\gamma^+$	0.298	0.261	0.277
$6_\gamma^+\to 4_\gamma^+$	0.336	0.352	0.595
$8_\gamma^+\to 6_\gamma^+$	0.314	0.549	0.814
$2_\gamma^+\to 0_g^+$	0.037	0.006	0.192
$2_\gamma^+\to 2_g^+$	0.303	0.303	0.055
$2_\gamma^+\to 4_g^+$	0.024	0.000	0.000
$4_\gamma^+\to 2_g^+$	0.002	0.004	0.274
$4_\gamma^+\to 4_g^+$	0.203	0.068	0.137
$4_\gamma^+\to 6_g^+$	0.018	0.000	0.000
$6_\gamma^+\to 4_g^+$	0.000	0.002	0.357
$6_\gamma^+\to 6_g^+$	0.171	0.042	0.171

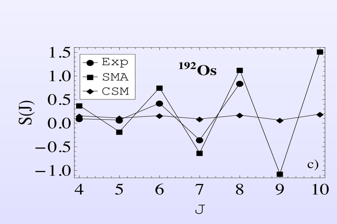


FIGURE 2: Excitation energies, given in keV, for ground, beta and gamma bands and E2 transition probabilities of ^{192}Os, calculated with SMA and CSM, are compared with the corresponding experimental data [16,17]. Experimental and theoretical staggering $S(J)$.

Tables: Excitation energies (left side), given in keV units, for ground, beta and gamma bands and the reduced E2 transition probabilities of ^{188}Os, calculated with SSA, CSM, X(5), ISW and D models. The experimental data are taken from Ref. [18].

^{188}Os	Exp.	X(5)	ISW	D	SSA	CSM
2_g^+	155	179	179	151	152	150
4_g^+	478	519	519	479	476	468
6_g^+	940	970	970	945	935	934
8_g^+	1515	1516	1516	1512	1501	1535
10_g^+	2170	2149	2150	2156	2154	2264
12_g^+	2856	2867	2868	2860	2877	3116
0_β^+	1086	1009	1007	1120	1063	1164
2_β^+	1305	1331	1328	1270	1330	1305
4_β^+		1910	1907	1599	1808	1621
6_β^+		2636	2632	2064	2421	2096
8_β^+		3474	3470	2632	3132	2717
10_β^+		4412	4407	3276	3920	3475
2_γ^+	633	631	631	627	641	665
3_γ^+	790	786	785	773	791	790
4_γ^+	966	972	971	959	969	956
5_γ^+	1181	1185	1185	1180	1172	1157
6_γ^+	1425	1423	1423	1432	1434	1399
7_γ^+	1686	1685	1684	1709	1674	1669
8_γ^+		1969	1969	2009	2008	1983
9_γ^+		2275	2275	2329	2273	2318
10_γ^+		2602	2603	2666	2670	2701
$r.m.s.$ [keV]		27	27	16	13	36

B(E2)(W.u.)	Exp.	X(5)	ISW	D	SSA	CSM
$2_g^+\to 0_g^+$	79^{+2}_{-2}	74	72	79	82	42
$4_g^+\to 2_g^+$	133^{+8}_{-8}	118	115	121	123	87
$6_g^+\to 4_g^+$	138^{+8}_{-8}	147	144	147	145	125
$8_g^+\to 6_g^+$	161^{+11}_{-11}	169	166	174	162	161
$10_g^+\to 8_g^+$	188^{+25}_{-25}	187	184	203	178	195
$0_\beta^+\to 2_g^+$	$0.95^{+0.08}_{-0.08}$	47	48	33	21	0.95
$0_\beta^+\to 2_\gamma^+$	$4.3^{+0.5}_{-0.5}$	5.2	5.2	1.9	1.5	44
$4_\gamma^+\to 2_\gamma^+$	47^{+10}_{-10}	47	50	52	56	14
$4_\gamma^+\to 3_\gamma^+$	320^{+120}_{-120}	112	117	120	132	43
$6_\gamma^+\to 4_\gamma^+$	70^{+30}_{-30}	107	111	114	118	31
$2_\gamma^+\to 0_g^+$	$5^{+0.6}_{-0.6}$	8.4	10.9	10.8	9.9	5
$2_\gamma^+\to 2_g^+$	16^{+2}_{-2}	13	17	16	14	10.4
$2_\gamma^+\to 4_g^+$	34^{+5}_{-5}	0.65	0.85	0.80	0.73	1.4
$4_\gamma^+\to 2_g^+$	$1.29^{+0.19}_{-0.19}$	5.7	7.1	6.7	6.1	1.7
$4_\gamma^+\to 4_g^+$	19^{+3}_{-3}	18	23	20	19	10.7
$4_\gamma^+\to 6_g^+$	16^{+7}_{-7}	2	2	2	2	5
$6_\gamma^+\to 4_g^+$	$0.21^{+0.11}_{-0.11}$	5.3	6.4	5.8	5.3	0.9
$6_\gamma^+\to 6_g^+$	>9.4	21	25	23	20	8.3

Comparing the results for ^{188}Os presented in Tables, we can see that the best agreement with experimental data, for both energy spectrum and *E2* transition probabilities, is obtained with SSA.

Conclusions

The main contributions of this work are:

- SSA and SMA represent realistic tools for the description of X(5) candidate nuclei and of triaxial nuclei with equilibrium shapes close to $\gamma_0=\pi/6$.
- A salient feature of our investigations consists of that the Mathieu and spheroidal functions are periodic, defined on bounded intervals and normalized to unity with the integration measure $|\sin 3\gamma|d\gamma$, preserving in this way the hermiticity of the initial γ Hamiltonian.
- Highlighting the fact that the Coherent State Model works very well also for nuclei being in critical points of the shape phase transitions.

References

[1] A. Bohr, Mat. Fys. Medd. Dan. Vid. Selsk. **26** (1952) no.14; A.Bohr and B.Mottelson, Mat. Fys. Medd. Dan. Vid. Selsk. **27** (1953) no. 16; **[2]** L. Fortunato , Eur. Phys. J. A **26**, s01, 1-30 (2005). **[3]** A. Faessler and W. Greiner, Z. Phys. **168** (1962) 425; **170** (1962) 105; **177** (1964) 190; A. Faessler, W. Greiner and R. Sheline, Nucl. Phys. **70** (1965) 33. **[4]** F. Iachello, Phys. Rev. Lett. **85** (2000) 3580. **[5]** F. Iachello, Phys. Rev. Lett. **87** (2001) 052502. **[6]** F. Iachello, Phys. Rev. Lett. **91** 132502 (2003). **[7]** D. Bonatsos, D. Lenis, D. Petrellis, P.A. Terziev,Phys. Lett. **B 588**, 172 (2004). **[8]** A. A. Raduta and P. Buganu, Phys. Rev. C **83**, 034313 (2011). **[9]** P. Buganu, A. A. Raduta and A. Faessler, J. Phys. G: Nucl. Part. Phys. **39**, 025103 (2012). **[10]** P. Buganu and A. A. Raduta, Rom. Journ. Phys. **57**, 1103-1112 (2012). **[11]** A. A. Raduta and P. Buganu, J. Phys. G: Nucl. Part. Phys. **40**, 025108 (2013). **[12]** L. Wilets and M. Jean, Phys. Rev.**102**, 788 (1956). **[13]** L. Fortunato, Eur. J. Phys. **A 26**, s01, 1 (2005). **[14]** A. A. Raduta, A. C. Gheorghe, P. Buganu and A. Faessler, Nucl. Phys. **A 819**, 46 (2009). **[15]** A. A. Raduta, V. Ceausescu, A. Gheorghe and R. M. Dreizler, Phys. Lett. **99B** (1981) 444; Nucl. Phys. **A381** (1982) 253. **[16]** C. Y. Wu *et al.*,Nucl. Phys. A **A 607** (1996) 178. **[17]** Coral M. Baglin,Nuclear Data Sheets **84**, 717 (1998). **[18]** Balraj Singh, Nuclear Data Sheets **95**, 387 (2002).

INTERNATIONAL NUCLEAR PHYSICS CONFERENCE, FLORENCE (ITALY), 2-7 June 2013