

Description of the $2\nu\nu\beta\beta$ decay within a gauge restored of a fully renormalized pnQRPA approach

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Abstract

A many body Hamiltonian involving the mean field for a projected spherical single particle basis, the pairing interactions for alike nucleons, the dipole-dipole proton-neutron interactions in the particle-hole (ph) channel and the ph dipole pairing potential, is treated by the projected gauge of fully renormalized proton-neutron quasiparticle random phase approximation (PGFRpnQRPA) approach. The resulting wave functions and energies for the mother and the daughter nuclei are used to calculate the $2\nu\beta\beta$ decay rate and the process half life. For illustration, the formalism is applied for the decays $^{100}\text{Mo}\rightarrow^{100}\text{Ru}$ and $^{116}\text{Cd}\rightarrow^{116}\text{Sn}$. The results are in good agreement with the corresponding experimental data. The Ikeda sum rule (ISR) is obeyed.

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I. INTRODUCTION

Double beta decay is one of the most exciting topic of nuclear physics because of the possible existence of neutrinoless $\beta\beta$ decay modes, which are very searched since they are connecting to the neutrino mass and neutrino nature. The calculation of the nuclear matrix elements for these decay modes is a challenge in the study of $\beta\beta$ decay process; however, this calculations implies the calculation not only of Gamow-Teller transitions but also of Fermi and tensor transitions.

The $2\nu\beta\beta$ process is interesting by its own but is also very attractive because it constitutes a test for some of the nuclear matrix elements (m.e.) which are used for the process of $0\nu\beta\beta$ decay. Discovery of this process may provide an answer for the fundamental question, whether neutrino is a Majorana or a Dirac particle. The subject development is reflected be several review papers [1–7]. Our contribution described in this paper concerns the $2\nu\beta\beta$ process, which can be viewed as two consecutive and virtual single β^- decays. The formalism yielding closest results to the experimental data is the proton-neutron random phase approximation (pnQRPA) which includes the particle-hole (ph) and particle-particle (pp) [8] as independent two body interactions. The second leg of the $2\nu\beta\beta$ process is very sensitive to changing the relative strength of the later interaction, denoted hereafter by g_{pp} . It is worth mentioning that the two body interaction of ph type is repulsive while that of pp nature is attractive. Due to this feature there is a critical value for g_{pp} for which the first root of the pnQRPA equation vanishes. Actually, this is the signal that the $pnQRPA$ approach is no longer valid. Moreover, the g_{pp} value which corresponds to a transition amplitude which agrees with the corresponding experimental data is close to the mentioned critical value. That means that the result is not stable to adding corrections to the $pnQRPA$ picture. One improvement for the pnQRPA was achieved by one of us (AAR), in collaboration, in Refs.[9, 10], by using a boson expansion (BE) procedure. Another procedure of going beyond $pnQRPA$ is to renormalize the dipole two quasiparticle operators by replacing the scalar components of their commutators by their average values [11]. Such a renormalization is inconsistently achieved since the scattering operators are not renormalized. This lack of consistency was removed in Ref. [12, 13] where a fully renormalized pnQRPA (FRpnQRPA) is proposed.

Unfortunately, all higher pnQRPA procedures mentioned above have a common drawback

of violating the Ikeda sum rule (*ISR*) by an amount of about 20-30% [14]. It is believed that such a violation is caused by the gauge symmetry breaking. Consequently, a method of restoring this symmetry was formulated by the present authors in Ref. [15].

In this paper the results of Ref.[15] are improved in three respects: a) aiming at providing a unitary description of the process for the situations when the nuclei involved are spherical or deformed, here we use the projected spherical single particle basis defined in Ref.[16] and used for double beta decay in Refs.[17, 18]. b) the space of proton-neutron dipole configurations is split in three subspaces, one being associated to the single β^- , one to the β^+ process, and one spanned by the unphysical states. c) the correlations for the second leg of the process are mainly determined by the ph dipole-pairing term. A compact expression for the dispersion equation of energies is obtained from the linearized equations of motion of the basic transition operators corresponding to the two coupled processes. The numerical application is made for the $2\nu\beta\beta$ processes $^{100}\text{Mo} \rightarrow ^{100}\text{Ru}$ and $^{116}\text{Cd} \rightarrow ^{116}\text{Sn}$.

We shall describe the formalism and results according to the following plan. The single particle basis is briefly presented in Section II. The model Hamiltonian is given in Section III. The FRpnQRPA approach is discussed in Section IV, while the projected gauge of FRpnQRPA (PGFRpnQRPA) is the objective of Section V. The Gamow-Teller (GT) amplitude for the $2\nu\beta\beta$ process is given in Section VI. Numerical applications are shown in Section VII, while the final conclusions are drawn in Section VIII.

II. THE MODEL HAMILTONIAN

We suppose that the states describing the mother, the daughter and the intermediate odd-odd nuclei are described by a sole many-body Hamiltonian. Written in the second quantization with a projected spherical single particle basis [16, 17, 19–22], this is similar to the one used in Ref.[18], H' , with the difference that the proton-neutron dipole-dipole interaction in the particle-particle (pp) channel, H_{pp} , is replaced by an attractive term of dipole pairing type, H_{dp} :

$$\begin{aligned}
 H &= H' - H_{pp} + H_{dp}, \text{ with} \\
 H_{dp} &= -\chi_1 \sum_{\substack{pn;p' \\ n';\mu}} \left(\beta_{\mu}^{-}(pn)\beta_{-\mu}^{-}(p'n') + \beta_{-\mu}^{+}(p'n')\beta_{\mu}^{+}(pn) \right) (-1)^{1-\mu}, \quad (2.1)
 \end{aligned}$$

Here the same notation for the dipole operators β^\pm is used as in Ref.[18]. The reason for this modification consists of that the pp interaction is not effective within a gauge projection formalism. On the other hand one needs an attractive interaction in order to obtain a realistic description of the second leg of the double beta process. More arguments on this line will be given in Section IV.

Passing to the quasiparticle (qp) representation through the Bogoliubov-Valatin transformation:

$$a_{\tau IM}^\dagger = U_{\tau I} c_{\tau IM}^\dagger - s_{IM} V_{\tau I} c_{\tau I-M}, \quad s_{IM} = (-)^{I-M}, \quad \tau = p, n, \quad U_{\tau I}^2 + V_{\tau I}^2 = 1, \quad (2.2)$$

the first two terms of H are replaced by the independent quasiparticles term, $\sum E_{\tau I} a_{\tau IM}^\dagger a_{\tau IM}$, while the two-body dipole-dipole interactions are expressed in terms of the dipole two qp and the dipole qp density operators:

$$\begin{aligned} A_{1\mu}^\dagger(pn) &= \sum C_{m_p m_n}^{I_p I_n 1} a_{p I_p m_p}^\dagger a_{n I_n m_n}^\dagger, \quad A_{1\mu}(pn) = \left(A_{1\mu}^\dagger(pn) \right)^\dagger, \\ B_{1\mu}^\dagger(pn) &= \sum C_{m_p -m_n}^{I_p I_n 1} a_{p j_p m_p}^\dagger a_{n I_n m_n} (-)^{I_n - m_n}, \quad B_{1\mu}(pn) = \left(B_{1\mu}^\dagger(pn) \right)^\dagger. \end{aligned} \quad (2.3)$$

III. THE FULLY RENORMALIZED PNQRPA

In Ref.[12], we showed that all these operators can be renormalized as suggested by the commutation equations:

$$\begin{aligned} [A_{1\mu}(k), A_{1\mu'}^\dagger(k')] &\approx \delta_{k,k'} \delta_{\mu,\mu'} \left[1 - \frac{\hat{N}_n}{\hat{I}_n^2} - \frac{\hat{N}_p}{\hat{I}_p^2} \right], \\ [B_{1\mu}^\dagger(k), A_{1\mu'}^\dagger(k')] &\approx [B_{1\mu}^\dagger(k), A_{1\mu'}(k')] \approx 0, \\ [B_{1\mu}(k), B_{1\mu'}^\dagger(k')] &\approx \delta_{k,k'} \delta_{\mu,\mu'} \left[\frac{\hat{N}_n}{\hat{I}_n^2} - \frac{\hat{N}_p}{\hat{I}_p^2} \right], \quad k = (I_p, I_n). \end{aligned} \quad (3.1)$$

Indeed, denoting by $C_{I_p, I_n}^{(1)}$ and $C_{I_p, I_n}^{(2)}$ the averages of the right hand sides of (3.1) with the renormalized RPA vacuum state, the renormalized operators defined as

$$\bar{A}_{1\mu}(k) = \frac{1}{\sqrt{C_k^{(1)}}} A_{1\mu}, \quad \bar{B}_{1\mu}(k) = \frac{1}{\sqrt{|C_k^{(2)}|}} B_{1\mu}, \quad (3.2)$$

obey boson like commutation relations:

$$\begin{aligned} [\bar{A}_{1\mu}(k), \bar{A}_{1\mu'}^\dagger(k')] &= \delta_{k,k'} \delta_{\mu,\mu'}, \\ [\bar{B}_{1\mu}(k), \bar{B}_{1\mu'}^\dagger(k')] &= \delta_{k,k'} \delta_{\mu,\mu'} f_k, \quad f_k = \text{sign}(C_k^{(2)}). \end{aligned} \quad (3.3)$$

Further, these operators are used to define the phonon operator:

$$C_{1\mu}^\dagger = \sum_k \left[X(k) \bar{A}_{1\mu}^\dagger(k) + Z(k) \bar{D}_{1\mu}^\dagger(k) - Y(k) \bar{A}_{1-\mu}(k)(-)^{1-\mu} - W(k) \bar{D}_{1-\mu}(k)(-)^{1-\mu} \right], \quad (3.4)$$

where $\bar{D}_{1\mu}^\dagger(k)$ is equal to $\bar{B}_{1\mu'}^\dagger(k')$ or $\bar{B}_{1\mu}(k)$ depending on whether f_k is + or -. The phonon amplitudes are determined by the equations:

$$[H, C_{1\mu}^\dagger] = \omega C_{1\mu}^\dagger \quad [C_{1\mu}, C_{1\mu'}^\dagger] = \delta_{\mu\mu'}. \quad (3.5)$$

Interesting properties for these equations and their solutions are discussed in our previous publications [12, 13].

IV. THE PROJECTED GAUGE OF THE FULLY RENORMALIZED PNQRPA

The renormalized ground state, i.e., the vacuum state for the phonon operator defined by the FRpnQRPA approach, is a superposition of components describing the neighboring nuclei $(N-1, Z+1)$, $(N+1, Z-1)$, $(N+1, Z+1)$, $(N-1, Z-1)$. The first two components conserve the total number of nucleons ($N+Z$) but violates the third component of isospin, T_3 . By contrast, the last two components violates the total number of nucleons but preserve T_3 . Actually, the last two components contribute to the violation of the *ISR*. However, one can construct linear combinations of the basic operators $A^\dagger, A, B^\dagger, B$ which excite the nucleus (N, Z) to the nuclei $(N-1, Z+1)$, $(N+1, Z-1)$, $(N+1, Z+1)$, $(N-1, Z-1)$, respectively. These operators are:

$$\begin{aligned} \mathcal{A}_{1\mu}^\dagger(pn) &= U_p V_n A_{1\mu}^\dagger(pn) + U_n V_p A_{1,-\mu}(pn)(-)^{1-\mu} + U_p U_n B_{1\mu}^\dagger(pn) - V_p V_n B_{1,-\mu}(pn)(-)^{1-\mu}, \\ \mathcal{A}_{1\mu}(pn) &= U_p V_n A_{1\mu}(pn) + U_n V_p A_{1,-\mu}^\dagger(pn)(-)^{1-\mu} + U_p U_n B_{1\mu}(pn) - V_p V_n B_{1,-\mu}^\dagger(pn)(-)^{1-\mu}, \\ \mathbf{A}_{1\mu}^\dagger(pn) &= U_p U_n A_{1\mu}^\dagger(pn) - V_p V_n A_{1,-\mu}(pn)(-)^{1-\mu} - U_p V_n B_{1\mu}^\dagger(pn) - V_p U_n B_{1,-\mu}(pn)(-)^{1-\mu}, \\ \mathbf{A}_{1\mu}(pn) &= U_p U_n A_{1\mu}(pn) - V_p V_n A_{1,-\mu}^\dagger(pn)(-)^{1-\mu} - U_p V_n B_{1\mu}(pn) - V_p U_n B_{1,-\mu}^\dagger(pn)(-)^{1-\mu}. \end{aligned}$$

Indeed, in the particle representation these operators have the expressions:

$$\mathcal{A}_{1\mu}^\dagger(pn) = -[c_p^\dagger c_n^-]_{1\mu}, \quad \mathcal{A}_{1\mu}(pn) = -[c_p^\dagger c_n^-]_{1\mu}^\dagger, \quad (4.1)$$

$$\mathbf{A}_{1\mu}^\dagger(pn) = [c_p^\dagger c_n^\dagger]_{1\mu}, \quad \mathbf{A}_{1\mu}(pn) = [c_p^\dagger c_n^\dagger]_{1\mu}^\dagger. \quad (4.2)$$

In terms of the new operators the many body Hamiltonian is:

$$\begin{aligned} H = & \sum_{\tau jm} E_{\tau j} a_{\tau jm}^\dagger a_{\tau jm} + 2\chi \sum_{pn,p'n';\mu} \sigma_{pn;p'n'} \mathcal{A}_{1\mu}^\dagger(pn) \mathcal{A}_{1\mu}(p'n') - \chi_1 \sum_{\substack{pn;p' \\ n';\mu}} \sigma_{pn;p'n'} \\ & \times \left(\mathcal{A}_{1\mu}^\dagger(pn) \mathcal{A}_{1,-\mu}^\dagger(p'n') + \mathcal{A}_{1,-\mu}(p'n') \mathcal{A}_{1\mu}(pn) \right) (-)^{1-\mu}, \\ & \sigma_{pn;p'n'} = \frac{2}{3\hat{I}_n \hat{I}_{n'}} \langle I_p || \sigma || I_n \rangle \langle I_{p'} || \sigma || I_{n'} \rangle. \end{aligned} \quad (4.3)$$

Here $E_{\tau I}$ denotes the quasiparticle energy. If instead of the dipole ph pairing interaction we consider the proton-neutron two-body pp interaction, then the third term of H would be:

$$H_3 = -2\chi_1 \sum_{\substack{pn;p' \\ n';\mu}} \sigma_{pn;p'n'} \mathbf{A}_{1\mu}^\dagger(pn) \mathbf{A}_{1\mu}(p'n'). \quad (4.4)$$

Since we are interested in describing the harmonic modes which preserve the total number of nucleons, the term H_3 would not contribute at the *RPA* level. Indeed, this term defines a deuteron type excitation, and consequently modifies the total number of nucleons.

The equations of motion of the operators defining the phonon operator are determined by the commutation relations:

$$[\mathcal{A}_{1\mu}(pn), \mathcal{A}_{1\mu'}^\dagger(p'n')] \approx \delta_{\mu,\mu'} \delta_{j_p,j_{p'}} \delta_{j_n,j_{n'}} \left[U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{\hat{j}_n^2} \hat{N}_n - \frac{U_p^2 - V_p^2}{\hat{j}_p^2} \hat{N}_p \right]. \quad (4.5)$$

The average of the r.h. side of this equation with the *PGFRpnQRPA* vacuum state is denoted by:

$$D_1(pn) = U_p^2 - U_n^2 + \frac{1}{2I_n + 1} (U_n^2 - V_n^2) \langle \hat{N}_n \rangle - \frac{1}{2I_p + 1} (U_p^2 - V_p^2) \langle \hat{N}_p \rangle. \quad (4.6)$$

The equations of motion show that the two qp energies are renormalized too:

$$E^{ren}(pn) = E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2). \quad (4.7)$$

The space of pn dipole states, \mathcal{S} , is written as a sum of three subspaces defined as:

$$\mathcal{S}_+ = \{(p, n) | D_1(pn) > 0, E^{ren}(pn) > 0\}, \quad \mathcal{S}_- = \{(p, n) | D_1(pn) < 0, E^{ren}(pn) < 0\},$$

$$\mathcal{S}_{sp} = \mathcal{S} - (\mathcal{S}_+ + \mathcal{S}_-),$$

$$\mathcal{N}_\pm = \dim(\mathcal{S}_\pm), \quad \mathcal{N}_{sp} = \dim(\mathcal{S}_{sp}),$$

$$\mathcal{N} = \mathcal{N}_+ + \mathcal{N}_- + \mathcal{N}_{sp}. \quad (4.8)$$

The third line of the above equations specify the dimensions of these subspaces. In \mathcal{S}_+ one defines the renormalized operators:

$$\bar{\mathcal{A}}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{D_1(pn)}} \mathcal{A}_{1\mu}^\dagger(pn), \quad \bar{\mathcal{A}}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}} \mathcal{A}_{1\mu}(pn), \quad (4.9)$$

while in \mathcal{S}_- the renormalized operators are:

$$\bar{\mathcal{F}}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{|D_1(pn)|}} \mathcal{A}_{1\mu}(pn), \quad \bar{\mathcal{F}}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}} \mathcal{A}_{1\mu}^\dagger(pn). \quad (4.10)$$

Indeed, the operator pairs $\mathcal{A}_{1\mu}, \mathcal{A}_{1\mu}^\dagger$ and $\mathcal{F}_{1\mu}, \mathcal{F}_{1\mu}^\dagger$ satisfy commutation relations of boson type. An RPA treatment within \mathcal{S}_{sp} would yield either vanishing or negative energies. The corresponding states are therefore spurious.

The equations of motion for the renormalized operators read:

$$\begin{aligned} [H, \bar{\mathcal{A}}_{1\mu}^\dagger(pn)] &= E^{ren}(pn) \bar{\mathcal{A}}_{1\mu}^\dagger(pn) + 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1\mu}^\dagger(p_1 n_1) - 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1\mu}^\dagger(p_1 n_1) \\ &\quad + 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1-\mu}(-1)^{1-\mu}(p_1 n_1) - 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1 n_1), \\ [H, \bar{\mathcal{F}}_{1\mu}^\dagger(pn)] &= |E^{ren}(pn)| \bar{\mathcal{F}}_{1\mu}^\dagger(pn) + 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1\mu}^\dagger(p_1 n_1) - 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1\mu}^\dagger(p_1 n_1) \\ &\quad + 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1 n_1) - 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1-\mu}(p_1 n_1) (-1)^{1-\mu}, \\ [H, \bar{\mathcal{A}}_{1\mu}(pn)] &= -E^{ren}(pn) \bar{\mathcal{A}}_{1\mu}(pn) - 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1\mu}(p_1 n_1) + 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1\mu}(p_1 n_1) \\ &\quad - 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1,-\mu}^\dagger(p_1 n_1) (-1)^{1-\mu} + 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1-\mu}^\dagger(p_1 n_1) (-1)^{1-\mu}, \\ [H, \bar{\mathcal{F}}_{1\mu}(pn)] &= -|E^{ren}(pn)| \bar{\mathcal{F}}_{1\mu}(pn) - 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1\mu}(p_1 n_1) + 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1\mu}(p_1 n_1) \\ &\quad - 2\chi \sum_{(p_1 n_1) \in \mathcal{S}_+} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{A}}_{1,-\mu}^\dagger(p_1 n_1) (-1)^{1-\mu} + 2\chi_1 \sum_{(p_1 n_1) \in \mathcal{S}_-} \sigma_{pn; p_1 n_1}^{(1)} \bar{\mathcal{F}}_{1-\mu}^\dagger(p_1 n_1) (-1)^{1-\mu}, \end{aligned} \quad (4.11)$$

where:

$$\sigma_{pn; p_1 n_1}^{(1)} = \frac{2}{\hat{I}_n} \langle p || \sigma || n \rangle |D_1(pn)|^{1/2} \frac{2}{\hat{I}_{n_1}} \langle p_1 || \sigma || n_1 \rangle |D_1(p_1 n_1)|^{1/2} \equiv T_{pn} T_{p_1 n_1}. \quad (4.12)$$

The phonon operator is defined as:

$$\Gamma_{1\mu}^\dagger = \sum_{k=(pn)} [X(k) \bar{\mathcal{A}}_{1\mu}^\dagger(k) + Z(k) \bar{\mathcal{F}}_{1\mu}^\dagger(k) - Y(k) \bar{\mathcal{A}}_{1-\mu}(k) (-1)^{1-\mu} - W(k) \bar{\mathcal{F}}_{1-\mu}(k) (-1)^{1-\mu}]. \quad (4.13)$$

with the amplitudes determined by the equations:

$$[H, \Gamma_{1\mu}^\dagger] = \omega \Gamma_{1\mu}^\dagger, \quad [\Gamma_{1\mu}, \Gamma_{1\mu'}^\dagger] = \delta_{\mu, \mu'}. \quad (4.14)$$

Thus the phonon amplitudes are obtained by solving the PGFRpnQRPA equations:

$$\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ -B_{11} & -B_{12} & -A_{11} & -A_{12} \\ -B_{21} & -B_{22} & -A_{21} & -A_{22} \end{pmatrix} \begin{pmatrix} X(pn) \\ Z(pn) \\ Y(pn) \\ W(pn) \end{pmatrix} = \omega \begin{pmatrix} X(p_1 n_1) \\ Z(p_1 n_1) \\ Y(p_1 n_1) \\ W(p_1 n_1) \end{pmatrix}. \quad (4.15)$$

where the following notations have been used:

$$\begin{aligned} (A_{11}) &= E^{ren}(pn) \delta_{pn; p_1 n_1} + 2\chi \sigma_{p_1 n_1; pn}^{(1)T}, \\ (A_{12}) &= -2\chi_1 \sigma_{p_1 n_1; pn}^{(1)T}, \\ (B_{11}) &= -2\chi_1 \sigma_{p_1 n_1; pn}^{(1)T}, \\ (B_{12}) &= 2\chi \sigma_{p_1 n_1; pn}^{(1)T}, \\ (A_{21}) &= -2\chi_1 \sigma_{p_1 n_1; pn}^{(1)T}, \\ (A_{22}) &= |E^{ren}(pn)| \delta_{pn; p_1 n_1} + 2\chi \sigma_{p_1 n_1; pn}^{(1)T}, \\ (B_{21}) &= 2\chi \sigma_{p_1 n_1; pn}^{(1)T}, \\ (B_{22}) &= -2\chi_1 \sigma_{p_1 n_1; pn}^{(1)T}. \end{aligned} \quad (4.16)$$

Here the index T suggests the fact that the matrix is transposed. Matrix dimension for A_{11} and B_{11} is $\mathcal{N}_+ \times \mathcal{N}_+$ while for A_{22} and B_{22} is $\mathcal{N}_- \times \mathcal{N}_-$. The off diagonal sub-matrices A_{12} and B_{12} have the dimension $\mathcal{N}_+ \times \mathcal{N}_-$ while A_{21} and B_{21} are of the $\mathcal{N}_- \times \mathcal{N}_+$ type.

In order to solve Eqs.(4.15) we need to know $D_1(pn)$ and, therefore, the averages of the qp 's number operators, \hat{N}_p and \hat{N}_n . These are written first in particle representation and then the particle number conserving term is expressed as a linear combination of $\mathcal{A}^\dagger \mathcal{A}$ and $\mathcal{F}^\dagger \mathcal{F}$ chosen such that their commutators with $\mathcal{A}^\dagger, \mathcal{A}$ and $\mathcal{F}^\dagger, \mathcal{F}$ are preserved. The final result is:

$$\begin{aligned} \langle \hat{N}_p \rangle &= V_p^2 (2I_p + 1) + 3(U_p^2 - V_p^2) \left(\sum_{\substack{n', k \\ (p, n') \in \mathcal{S}_+}} D_1(p, n') (Y_k(p, n'))^2 - \sum_{\substack{n', k \\ (p, n') \in \mathcal{S}_-}} D_1(p, n') (W_k(p, n'))^2 \right), \\ \langle \hat{N}_n \rangle &= V_n^2 (2I_n + 1) + 3(U_n^2 - V_n^2) \left(\sum_{\substack{p', k \\ (p', n) \in \mathcal{S}_+}} D_1(p', n) (Y_k(p', n))^2 - \sum_{\substack{p', k \\ (p', n) \in \mathcal{S}_-}} D_1(p', n) (W_k(p', n))^2 \right). \end{aligned} \quad (4.17)$$

Eqs. (4.15), (4.17) and (4.6) are to be simultaneously considered and solved iteratively. It is worth mentioning that using the quasiparticle representation for the basic operators $\mathcal{A}_{1\mu}^\dagger, \mathcal{F}_{1\mu}^\dagger, \mathcal{A}_{1,-\mu}(-1)^{1-\mu}, \mathcal{F}_{1,-\mu}(-1)^{1-\mu}$, one obtains for $\Gamma_{1\mu}^\dagger$ an expression which involves the scattering pn operators. Thus, the present approach is, indeed, the *PGFRpnQRPA*.

It is worth noting that the compatibility condition for the *PGpnQRPA* equations (4.15) can be written in a compact form as a dispersion equation for the excitation energies. This is explicitly given in Appendix A.

V. THE $2\nu\beta\beta$ PROCESS

The formalism presented above was used to describe the $2\nu\beta\beta$ process. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q-value of the double beta decay process

$$\Delta E = m_e c^2 + \frac{1}{2} Q_{\beta\beta}, \quad (5.1)$$

the reciprocal value of the $2\nu\beta\beta$ half life can be factorized as:

$$(T_{1/2}^{2\nu\beta\beta})^{-1} = F |M_{GT}(0_i^+ \rightarrow 0_f^+)|^2, \quad (5.2)$$

where F is an integral on the phase space, independent of the nuclear structure, while M_{GT} stands for the Gamow-Teller transition amplitude and has the expression :

$$M_{GT} = \sqrt{3} \sum_{k,k'} \frac{i \langle 0 || \beta_i^+ || 1_k \rangle_{ii} \langle 1_k | 1_{k'} \rangle_{ff} \langle 1_{k'} || \beta_f^+ || 0 \rangle_f}{E_k + \Delta E + E_{1^+}}. \quad (5.3)$$

In the above equation, the denominator consists of three terms: a) ΔE , which was already defined, b) the average value of the k-th *PDFRpnQRPA* energy normalized to the particular value corresponding to k=1, and c) the experimental energy for the lowest 1^+ state. The indices carried by the β^+ operators indicate that they act in the space spanned by the *PGFRpnQRPA* states associated to the initial (i) or final (f) nucleus, respectively. The overlap m.e. of the single phonon states in the initial and final nuclei respectively, are calculated within *PGFRpnQRPA*. In Eq.(5.3), the Rose convention for the reduced m.e. is used [24].

Note that if we restrict the pn space to \mathcal{S}_+ and moreover the χ_1 interaction is missing, M_{GT} vanishes due to the second leg of the transition. Indeed, the m.e. associated to the daughter

nucleus is of the type ${}_f\langle 0|(c_n^\dagger c_p)_{1\mu}(c_n^\dagger c_p)_{1\mu}|o\rangle_f$, which is equal to zero due to the Pauli principle restriction. In this case the equations of motion are of Tam Dankoff type and therefore the ground state correlations are missing. In order to induce the necessary correlations we have either to extend the formalism in the space \mathcal{S}_- , or to allow the ph excitations to interact via a pairing like force. Here the two effects are simultaneously considered. Also, we remark that the operator $\bar{\mathcal{A}}_{1\mu}^\dagger$ plays the role of a β^- transition operator, while when $\bar{\mathcal{F}}_{1\mu}^\dagger$ is applied on the ground state of the daughter nucleus, it induces a β^+ transition. Therefore, the $\beta\beta$ decay cannot be described by considering the β^- transition alone.

VI. NUMERICAL APPLICATION

For illustration, we present the results for the transitions $^{100}\text{Mo}\rightarrow^{100}\text{Ru}$ and $^{116}\text{Cd}\rightarrow^{116}\text{Sn}$. For these cases the energy corrections involved in Eq.(5.3) are:

$$\begin{aligned}\Delta E(^{100}\text{Mo}) &= 2.026\text{MeV}, & E_{1+}(^{100}\text{Tc}) &= 0.0\text{MeV}. \\ \Delta E(^{116}\text{Cd}) &= 1.916\text{MeV}, & E_{1+}(^{116}\text{In}) &= 0.0\text{MeV}.\end{aligned}\tag{6.1}$$

The parameters defining the single particle energies are those of the spherical shell model, the deformation parameter d and the parameter k relating the quadrupole coordinate with the quadrupole bosons as shown in Eq.(2.2).

These are fixed as described in Ref.[18]. The core system for the two decays is defined by $(Z, N) = (20, 20)$. Labeling the states according to their energies ordering, the single particle space is defined by the indices interval $[11, 55]$ and $[11, 65]$, respectively. The dimensions for the spaces $(\mathcal{S}_+, \mathcal{S}_-, \mathcal{S})$ are $(137, 1, 163)$ and $(139, 2, 175)$ for ^{100}Mo and ^{100}Ru , while for the mother and daughter nuclei of the decay $^{116}\text{Cd}\rightarrow^{116}\text{Sn}$, they are $(189, 0, 219)$ and $(182, 4, 219)$, respectively. For the both processes considered here eight iterations were necessary in order that the iteration process reaches the convergence. The strength of the dipole pn two-body interaction is usually taken to be

$$\chi = \frac{5.2}{A^{0.7}}\text{MeV}.\tag{6.2}$$

This expression was obtained by fitting the positions of the GT resonances in ^{40}Ca , ^{90}Zr and ^{208}Pb [25]. This expression provides for χ the values 0.207 and 0.187 for ^{100}Mo and ^{116}Cd , respectively. These values yield for the $\log ft$ values of the intermediate odd-odd nuclei

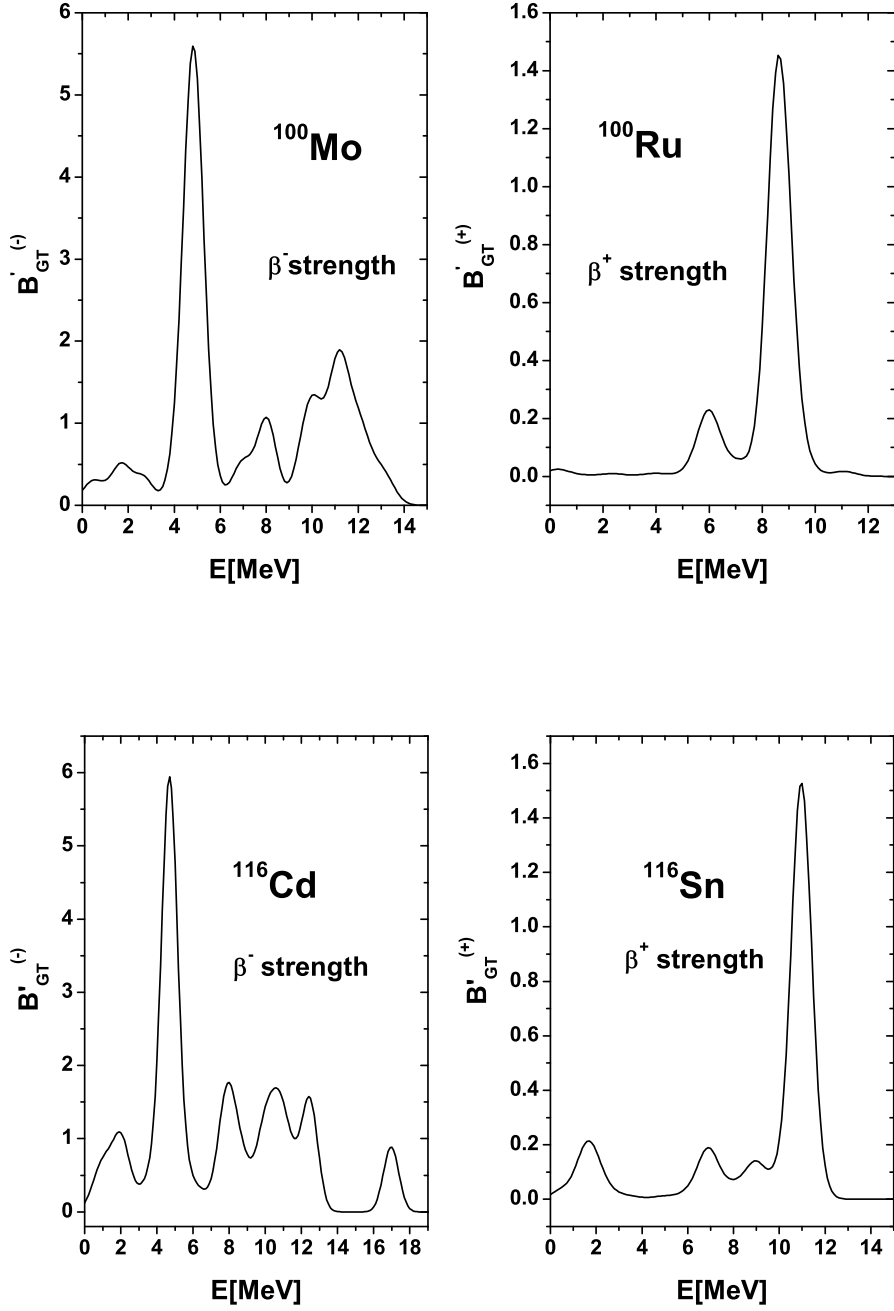


FIG. 1: One third of the single β^- strength for the mother nucleus, ^{100}Mo (upper-left panel) and ^{116}Cd (bottom-left), and one third of the β^+ strength for the daughter nucleus, ^{100}Ru (upper-right panel) and ^{116}Sn (bottom-right panel), folded by a Gaussian function with a width of 1 MeV, are plotted as functions of the corresponding energies yielded by the present formalism. Note that for a given nucleus the difference $B'_{GT}{}^{(-)} - B'_{GT}{}^{(+)}$ is to be compared with the reduced *ISR* value i.e., $N - Z$.

	d	k	G_p [MeV]	G_n [MeV]	ISR	$\log ft$	χ [MeV]	χ_1 [MeV]
^{100}Mo	-1.5	5.5	0.18	0.288	15.995	$^{100}\text{Mo} \xrightarrow{\beta^+/EC} ^{100}\text{Tc}$ 4.45 $^{+0.18}_{-0.30}$	0.232	1.406
^{100}Ru	-0.6	5.5	0.15	0.255	12.002	$^{100}\text{Tc} \xrightarrow{\beta^-} ^{100}\text{Ru}$ 4.66 4.12	0.232	1.406
^{116}Cd	-1.8	12.	0.15	0.282	20.07	$^{116}\text{Cd} \xrightarrow{\beta^+/EC} ^{116}\text{In}$ 4.39 $^{+0.1}_{-0.15}$	0.2	1.308
^{116}Sn	-1.2	12.	0.12	0.2458	16.007	$^{116}\text{In} \xrightarrow{\beta^-} ^{116}\text{Sn}$ 4.662 4.08	0.2	1.308

TABLE I: The deformation parameter d , the pairing interaction strengths for protons (G_p) and neutrons (G_n) and the GT dipole interaction χ used in our calculations. We also give the parameter k relating the quadrupole coordinates and bosons (this is involved in the expression of the single particle energies) as well as the resulting $\log ft$ values characterizing the β^+/EC and β_- transitions of ^{100}Tc and ^{116}In , respectively. The results for $\log ft$ values, given in the right column, are compared to the experimental data from the left column.

results which deviate much from the corresponding experimental data. For this reason we fixed χ by fitting the $\log ft$ value characterizing the β^+/EC process of the intermediate nuclei. Results for χ obtained in this way are slightly different from those provided by the expression (6.2). The parameter χ_1 was fixed such that the $\log ft$ value characterizing the β^- decay of the intermediate odd-odd nuclei is close to the corresponding experimental data. The results obtained for the nuclei involved in the two processes are given in Table 1.

We note that the ISR is satisfied both for mother and daughter nuclei. In our calculation the ISR is sensitive to the dimension of the single particle basis. Indeed, choosing a basis of a smaller dimension the ISR would be underestimated. Another parameters which influence the magnitude of ISR are the pairing strengths. Indeed ISR is increasing by increasing G_n or decreasing G_p . Variation of χ and χ_1 does not affect much ISR . However by varying these parameters the single beta strengths are modified according to the interaction nature. The strength is transferred to the lower energy by the attractive interaction (χ_1) and pushed up by the repulsive one (χ).

Using these input data we calculated the distribution of the β^\pm strengths with the result

	M_{GT} [MeV ⁻¹]	$T_{1/2}$ [yr]				
		present	Exp.	Ref.[18]	Refs. [30,31]	Refs.[32,33,34]
¹⁰⁰ Mo→ ¹⁰⁰ Ru	0.221	8.79·10 ¹⁸	(8.0±0.16) · 10 ¹⁸ ^{a)} (0.115 ^{+0.03} _{-0.02}) · 10 ²⁰ ^{b)} 0.033 ^{+0.02} _{-0.01} · 10 ²⁰ ^{c,d)}	4.4·10 ¹⁸	2.9·10 ¹⁸ ³⁾	1.8·10 ¹⁸ ¹⁾
¹¹⁶ Cd→ ¹¹⁶ Sn	0.15998	2.02·10 ¹⁹	(3.2±0.3) · 10 ¹⁹ ^{e)}	3.86·10 ¹⁹	5.1·10 ¹⁹ ⁵⁾	8.3·10 ¹⁸ 3.75·10 ¹⁹ ⁶⁾

TABLE II: The Gamow-Teller amplitude for the $2\nu\beta\beta$ decay, in units of MeV⁻¹, and the corresponding half life ($T_{1/2}$), in units of yr , are listed for two ground to ground transitions. The experimental half lives for the transitions of ¹⁰⁰Mo (^{a)} Ref.[26], ^{b)} Ref.[27] ^{c)} Ref.[28], ^{d)} Ref.[29]), ¹¹⁶Cd (^{e)} Ref.[36]), are also given. In the second last column the results reported in Refs. [30] ³⁾ and [31] ⁵⁾ are given. Comparison is also made with the theoretical results from the last column reported in Refs.[32] (¹⁾),[33] (unmarked) and [34] (⁶⁾).

shown in Fig.1. The energy intervals where both distributions are large, contribute significantly to the double beta transition amplitude. In plotting the β^+ strength we ignored the values smaller than 0.01. The β^\mp strengths are fragmented among the $pnQRPA$ states, reflecting the fact that the single particle states are deformed. Note that the first peak for the β^- strength is the highest one while the one centered at higher energy has a large width and a fine substructure. The low energy peak is mainly determined by the attractive two-body interaction while the broad peak, i.e. the GT giant resonance, by the GT interaction. The β^+ strength is small in magnitude and less fragmented than the β^- strength. Also we note that the highest energy peak is the largest one.

Calculating first the GT transition amplitude and then the Fermi integral with $G_A = 1.254$, as in Ref.[4], we obtained the results given in Table II.

From Table II, one may see that the present approach provides for the half life of the double beta decay, values which are quite close to the experimental data. The results are compared with other theoretical calculations using different formalisms. Thus, in Ref.[18] a schematic Gamow-Teller proton-neutron interaction, in the particle-hole and particle-particle channels, is treated within a projected spherical single particle basis by a $pnQRPA$ approach. The results correspond to a large value of the parameter g_{pp} . Indeed, for the

two nuclei considered here, ^{100}Mo and ^{116}Cd , the parameters (χ, g_{pp}) are $(0.06, 1.6)$ and $(0.238, 1.68)$ respectively. In Refs.[30, 31], a realistic Bonn-interaction is treated by a higher $pnQRPA$ approach, using a single particle basis corresponding to the Woods-Saxon potential. Of course, the ISR is not obeyed given the fact that a higher $pnQRPA$ approach is used. The results shown in the last column were obtained using for proton-neutron interaction the Paris potential in both the ph and pp channel and a $pnQRPA$ formalism. It is worth mentioning that although within the $pnQRPA$ approach the ISR is obeyed the large value for the pp interaction strength raises the question whether the used formalism is still valid. On the other hand the higher $pnQRPA$ approach yielding the results shown in Table II on the column of Refs.[30, 31], leads necessarily to a violation of the ISR . We remark that these weak points are not present in the present formalism. In Ref.[37] the standard renormalized $pnQRPA$ was applied for calculating the rate of double beta transitions of many isotopes. Unfortunately, the list does not include the isotopes considered here and consequently we cannot compare the predictions presented here with those from the quoted reference. Another fully renormalized procedure which obeys the ISR was formulated in Ref.[38]. The dependence of single beta decay strengths on the particle-particle interaction was studied numerically in Ref.[39]. By contrast, in our case the pp interaction does not contribute to the ground state correlations. Actually for this reason we replaced it by a dipole pairing interaction. This interaction brings important contributions to the backward going RPA amplitudes. However, even if this two-body interaction is missing the ground state correlations would not vanish due to the presence of the amplitudes \mathcal{F} in the phonon operator expression. At its turn this is caused by the split of the pn dipole configurations in two orthogonal subspaces, one associated to the β^- transition while the other one to the β^+ decay. Also the ISR is obeyed. We added however the dipole pairing interaction in order to describe the $\log ft$ values associated to the single beta transitions of the intermediate odd-odd nuclei. The eigenvalue equations and the restrictions for the averaged quasiparticle number operators, in the two renormalization approaches, are different from each other.

Another experimental result concerns the summed strength for the β^- transition, denoted, conventionally, by $\sum B_{GT^-}$. The experimental value of this sum covers only a fraction of the sum rule limit of $3(N - Z)$. Therefore in order to get a fair comparison of the calculated and measured quantity, we have to quench the calculated strength by a factor 0.6 in order to account for the missing experimental strength. The results are presented in Table III.

Nucleus	¹⁰⁰ Mo	¹¹⁶ Cd
$0.6\sum B(GT)_-$	28.96	36.2
$\sum B(GT)_-^{Exp}$	26.69	32.7

TABLE III: The experimental summed strength for the β^- transition (second row) is compared with the corresponding theoretical values quenched by a factor 0.6, for the two considered double beta emitters.

The intermediate odd-odd nucleus, ¹⁰⁰Tc, can perform the transition β^+/EC , feeding ¹⁰⁰Mo, or the β^- transition to ¹⁰⁰Ru. The same is true for ¹¹⁶In which by means of a EC process may undergo to ¹¹⁶Cd while through a β^- decay can feed the nucleus ¹¹⁶Sn. The measured $\log ft$ values for these transitions, are given in Table 1. The corresponding theoretical results are obtained by means of the expression:

$$ft_{\mp} = \frac{6160}{[l\langle 1_1 || \beta^{\pm} || 0 \rangle_l g_A]^2}, \quad l = i, f. \quad (6.3)$$

In order to take account of the effect of distant states responsible for the "missing strength" in the giant GT resonance [4] we chose $g_A = 1.0$. As we already mentioned, these single beta transitions were used as to fix the strengths of the two body dipole-dipole interactions.

The matrix elements involved in the double beta transition amplitude of the two emitters, ¹⁰⁰Mo and ¹¹⁶Cd, have been experimentally investigated in Ref.[36]. The first matrix elements, describing the transitions $0_i^+ \rightarrow 1^+$, were obtained from the reactions ¹⁰⁰Mo(³He,t)¹⁰⁰Tc and ¹¹⁶Cd(³He,t)¹¹⁶In respectively, at $\theta_t \approx 0^0$, while the matrix elements for the $1^+ \rightarrow 0_f^+$ were derived from the known $\log ft$ value. For both cases the strength of the first β^- transition exhibits two bumps, one broad and called GTR1 while the second one less spread, located at lower energy and called GTR2. The centroid energies of the two resonances as well as the strength carried by each of them are compared with the theoretical results obtained with our approach, in Table IV.

From the mentioned Table we notice that the theoretical centroid energy for GTR1 is by about 2 MeV smaller than the experimental data. Moreover, the calculated strength carried by GTR1 is smaller than the corresponding experimental data. A reason for such a discrepancy might be the relative values for the attractive and repulsive dipole-dipole interaction intensities which favors the transfer of strength from the GTR states to the states from the resonance of GTR2 and those from around 5 MeV. It is an open question whether

Excited states	^{100}Tc						^{116}In					
	Ex[MeV]			B(GT)			Ex[MeV]			B(GT)		
	Exp.	Th.		Exp.	Th.		Exp.	Th.		Exp.	Th.	
GTR1	13.3	11.16		23.1± 3.8	15.63		14.5	12.37		25.8± 4.1	18.9	
GTR2	8.0	8.05		2.9±0.5	5.87		8.9	7.87		6.6±1.1	7.2	

TABLE IV: The strengths $B(\text{GT})$ of the single β^- transitions from the mother nuclei to the intermediate odd-odd nuclei excited in the states of the two components, GTR1 and GTR2, of the GT giant resonance are listed. The experimental (Exp.) and theoretical (Th.) values for the centroid energies of the two resonances are also specified.

these deviations could be washed out by a better fitting procedure or they constitute the price we have to pay for restoring the gauge symmetry. Note that the centroid for GTR2 of the $^{100}\text{Mo} \rightarrow ^{100}\text{Tc}$ transition is fairly well described by our approach. As for the transition $^{116}\text{Cd} \rightarrow ^{116}\text{In}$ the predicted centroid of GTR2 is 1 MeV lower in energy than the corresponding experimental data. The calculated strengths for GTR2 are larger than the corresponding experimental data. It is worth mentioning that the summed strength of the two resonances, GTR1 and GTR2, is reasonable close to the corresponding experimental data. A specific feature for our formalism consists of that states around the GTR1 centroid contribute to the peak seen in the strength distribution for the single β^+ decay of the daughter nuclei. Also for both nuclei it seems that the resonance around 5MeV for the β^- decay does not contribute at all to the double beta decay. Indeed, in this region the matrix element for the second leg of transition is almost vanishing. In this respect we notice that in the case of ^{116}Cd , a bunch of states below 2 MeV contribute to the double beta decay rate.

VII. CONCLUSIONS

Summarizing the results of this paper, one may say that restoring the gauge symmetry from the fully renormalized pnQRPA, one obtains a realistic description of the transition rate and moreover the *ISR* is obeyed. As shown in this paper, it seems that there is no need to include the *pp* interaction in the many body treatment of the process. Indeed, in the framework of a FRpnQRPA approach this interaction violates the total number of particle and consequently the gauge projection process makes it ineffective.

Note that the hypothesis saying that the double beta process consists of two successive single β^- decays requires a consistent description of the double beta and single beta processes. In our formalism, actually, this feature is met since the rate of the double beta decay and the log ft values associated to the single beta decays of the intermediate odd-odd nuclei are realistically described. Another issue which is worth to be mentioned refers to the chain of approximations of the many-body Hamiltonian. A measure of the consistency of all these approximations is the *ISR* which in our case is satisfied to a high accuracy.

The attractive interaction of ph dipole-pairing type is responsible for the ground state correlations. To a less extent these are caused by the \mathcal{F} components of the new phonon operator. The projection of gauge is essential for restoring the *ISR*. The gauge projection of the pnQRPA was previously achieved in Ref.[35] where the *ISR* is anyway satisfied within the unprojected picture. By contrast therein the effect of projection is small.

The *GPFRpnQRPA* equations consist of *FRpnQRPA* equation (5.12) supplemented by Eqs. (5.6) and (5.17) which must be simultaneously solved by an iteration procedure. Since the two-body interaction is a separable interaction, Eq.(5.2) may be replaced by the dispersion equation (A.1) for energies and Eqs.(A.3) for the four phonon amplitudes. An extensive study of all existent data, with the formalism described in this paper, will be presented in a subsequent work.

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VIII. APPENDIX A

The compatibility condition for the homogeneous system of equations determining the phonon amplitudes yields two dispersion equations for ω :

$$4(\chi^2 - \chi_1^2) [\mathcal{R}_-^+ - \mathcal{R}_+^-] [\mathcal{R}_+^+ - \mathcal{R}_-^-] + 4\chi \left[\sum_{\mathcal{S}_+} \frac{T_{pn}^2 E^{ren}(pn)}{\omega^2 - (E^{ren}(pn))^2} + \sum_{\mathcal{S}_-} \frac{T_{pn}^2 |E^{ren}(pn)|}{\omega^2 - |E^{ren}(pn)|^2} \right] = 1, \quad (\text{A.1})$$

with

$$\mathcal{R}_\pm^+ = \sum_{(p_1 n_1) \in \mathcal{S}_+} \frac{T_{p_1 n_1}^2}{\omega \pm E^{ren}(p_1 n_1)}, \quad \mathcal{R}_\pm^- = \sum_{(p_1 n_1) \in \mathcal{S}_-} \frac{T_{p_1 n_1}^2}{\omega \pm |E^{ren}(p_1 n_1)|}. \quad (\text{A.2})$$

The phonon amplitudes can be analytically determined. Indeed, the *GPFRpnQRPA* equations yield the following expressions for the four amplitudes;

$$\begin{aligned} X(pn) &= 2\frac{T_{pn}^2}{\omega - E^{ren}(pn)}(\chi\mathcal{X} - \chi_1\mathcal{Y}), \quad W(pn) = -2\frac{T_{pn}^2}{\omega + |E^{ren}(pn)|}(\chi\mathcal{X} - \chi_1\mathcal{Y}), \\ Z(pn) &= 2\chi\frac{T_{pn}^2}{\omega - |E^{ren}(pn)|}(\chi\mathcal{Y} - \chi_1\mathcal{X}), \quad Y(pn) = -2\chi\frac{T_{pn}^2}{\omega + E^{ren}(pn)}(\chi\mathcal{Y} - \chi_1\mathcal{X}) \end{aligned} \quad (\text{A.3})$$

The constant factors \mathcal{X} and \mathcal{Y} have the expressions:

$$\begin{aligned} \mathcal{X} &= \sum_{S_+} T_{p_1n_1} X(p_1n_1) + \sum_{S_-} T_{p_1n_1} W(p_1n_1), \\ \mathcal{Y} &= \sum_{S_+} T_{p_1n_1} Y(p_1n_1) + \sum_{S_-} T_{p_1n_1} Z(p_1n_1). \end{aligned} \quad (\text{A.4})$$

The two factors are related by:

$$\mathcal{Y} = \frac{1}{\chi_1} \left[\chi - \frac{1}{2(\mathcal{R}_-^+ - \mathcal{R}_+^-)} \right] \mathcal{X} \equiv \mathcal{U}\mathcal{X}. \quad (\text{A.5})$$

Finally the independent constant factor \mathcal{X} is determined from the normalization condition of the phonon operator. The result is:

$$\begin{aligned} \mathcal{X}^{-2} &= 4 \left[\sum_{S_+} \frac{T_{pn}^2}{(\omega - E^{ren}(pn))^2} - \sum_{S_-} \frac{T_{pn}^2}{(\omega + |E^{ren}(pn)|)^2} \right] (\chi - \chi_1\mathcal{U})^2 \\ &+ 4 \left[\sum_{S_-} \frac{T_{pn}^2}{(\omega - |E^{ren}(pn)|)^2} - \sum_{S_+} \frac{T_{pn}^2}{(\omega + E^{ren}(pn))^2} \right] (\chi_1 - \chi\mathcal{U})^2. \end{aligned} \quad (\text{A.6})$$

Having \mathcal{X} and \mathcal{Y} determined the phonon amplitudes are readily obtained by means of equations A.3.

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