On the geometry of Siegel-Jacobi domains

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November 16, 2010

Abstract

We study the holomorphic unitary representations of the Jacobi group based on Siegel-Jacobi domains. Explicit polynomial orthonormal bases of the Fock spaces based on the Siegel-Jacobi disk are obtained. The scalar holomorphic discrete series of the Jacobi group for the Siegel-Jacobi disk is constructed and polynomial orthonormal bases of the representation spaces are given.

M.S.C. 2000: 22E30, 20G05, 11F50, 12E10, 81R30

Key words: Jacobi group, Siegel-Jacobi domain, canonical automorphy factor, canonical kernel function, Fock representation, scalar holomorphic discrete series

1 Introduction

The Jacobi groups are semidirect products of appropriate semisimple real algebraic group of Hermitian type with Heisenberg groups [27], [13]. The semisimple groups are associated to Hermitian symmetric domains that are mapped into a Siegel upper half space by equivariant holomorphic maps [23]. The Jacobi groups are unimodular, nonreductive, algebraic groups of Harish-Chandra type. The Siegel-Jacobi domains are nonreductive symmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding [23], [13], [28] -[30].

The holomorphic irreducible unitary representations of the Jacobi groups based on Siegel-Jacobi domains have been constructed by Berndt, Böcherer, Schmidt, and Takase [9], [8], [25]-[27] with relevant topics: Jacobi forms, automorphic forms, spherical functions, theta functions, Hecke operators, and Kuga fiber varieties.

Some coherent state systems based on Siegel-Jacobi domains have been investigated in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, nuclear structure, and signal processing [12], [19], [24], [2]-[6].

This paper is organized as follows. In Section 2 we present explicit formulas for the canonical automorphy factors and kernel functions of the Jacobi groups and corresponding Siegel-Jacobi domains. In Section 3 we introduce a Fock space of holomorphic functions on the Siegel-Jacobi disk. We obtain explicit polynomial orthonormal bases for this space and the Fock spaces with inner products associated to points on the Siegel disk (Proposition 3.1 and Proposition 3.2). In Section 4 we construct the scalar holomorphic discrete series of the Jacobi group for the Siegel-Jacobi disk (Proposition 4.1). We give polynomial orthonormal bases of the representation spaces (Proposition 4.2). Finally, we discuss the link between the coherent state systems based on Siegel-Jacobi domains and the explicit kernel functions of representation spaces for Jacobi groups.

Notation. We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. $M_{mn}(\mathbb{F}) \approxeq \mathbb{F}^{mn}$ denotes the set of all $m \times n$ matrices with entries in the field $\mathbb{F}$. $M_{1n}(\mathbb{F})$ is identified with $\mathbb{F}^n$. Set $M_n(\mathbb{F}) = M_{nn}(\mathbb{F})$. For any $A \in M_{mn}(\mathbb{F})$, ${}^tA$ denotes the transpose matrix of A. For $A \in M_{mn}(\mathbb{C})$, $\bar{A}$ denotes the conjugate matrix of A and $A^\dagger = {}^t\bar{A}$. For $A \in M_n(\mathbb{C})$, the inequality $A > 0$ means that A is positive definite. The identity matrix of degree n is denoted by I_n. Let $\mathcal{O}(\mathfrak{D},W)$ denote the space of all W-valued holomorphic functions on the connected complex manifold $\mathfrak{D}$ equipped with the topology of uniform convergence on compact sets. Here W is a finite dimensional Hilbert space. Set $\mathcal{O}(\mathfrak{D})$ =$\mathcal{O}(\mathfrak{D},W)$ for $\dim W = 1$. In this paper we will use the words " unitary representation on a Hilbert space" to mean a continuous unitary representation on a complex separable Hilbert space.

2 Canonical automorphy factor and kernel function for Jacobi groups

We begin with the definition of the Jacobi group given in [9], [27], [13]. Let $\mathfrak{H}_n$ be the Siegel upper half space of degree n consisting of all symmetric matrices $\Omega \in M_n(\mathbb{C})$ with $\operatorname{Im}\Omega > 0$. Let $\mathrm{Sp}(n,\mathbb{R})$ be the symplectic group of degree n consisting of all matrices $\sigma \in M_{2n}(\mathbb{R})$ such that ${}^t\sigma J_n \sigma = J_n$, where

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{2.1}$$

and a, b, c, $d \in M_n(\mathbb{R})$. The group $\mathrm{Sp}(n,\mathbb{R})$ acts transitively on $\mathfrak{H}_n$ by $\sigma\Omega = (a\Omega + b)(c\Omega + d)^{-1}$, where $\sigma \in \mathrm{Sp}(n,\mathbb{R})$ and $\Omega \in \mathfrak{H}_n$.

Let G^s be a Zariski connected semisimple real algebraic group of Hermitian type. Let $\mathfrak{D}$ =G^s/K^s be the associated Hermitian symmetric domain, where K^s is a maximal compact subgroup of G. Suppose there exist a homomorphism $\rho : G^s \to \mathrm{Sp}(n,\mathbb{R})$ and a holomorphic map $\tau : \mathfrak{D} \to \mathfrak{H}_n$ such that $\tau(gz) = \rho(g)\tau(z)$ for all $g \in G^s$ and $z \in \mathfrak{D}$. The *Jacobi group* G^J is the semidirect product of G^s and the Heisenberg group $H[V]$ associated with the symplectic $\mathbb{R}$-space V and the nondegenerate alternating bilinear form $D : V \times V \to \mathcal{A}$, where $\mathcal{A}$ is the center of $H[V]$. The multiplication operation of $G^J \approx G^s \times V \times \mathcal{A}$ is defined by

$$gg' = (\sigma\sigma', \rho(\sigma)v' + v, \varkappa + \varkappa' + \frac{1}{2}D(v, \rho(\sigma)v'), \tag{2.2}$$

where $g = (\sigma, v, \varkappa) \in G^J$, $g' = (\sigma', v', \varkappa') \in G^J$.

The *Jacobi-Siegel domain* associated to the Jacobi group G^J is defined by $\mathfrak{D}^J = \mathfrak{D}\times\mathbb{C}^N \cong G^J/(K^s \times \mathcal{A})$, where $\dim V = 2N$.

Let w_0 be a fixed element of $\mathfrak{D}$ and let $I_{\tau(w_0)}$ be the complex structure on V corresponding to $\tau(w_0) \in \mathfrak{H}_n$. Let $V_{\mathbb{C}} = V_+ \oplus V_-$ be the complexification of V, where $V_\pm$ consists of all $v \in V_{\mathbb{C}}$ such that $I_{\tau(w_0)}v = \pm iv$. Then $w \in \mathfrak{D}$ and $v \in V_{\mathbb{C}}$ determine the element $v_w = v_+ - \tau(w)v_-$ of V_+, where $v = v_+ + v_-$, $v_\pm \in V_\pm$.

G^J is an algebraic group of Harish-Chandra type [27], [13], [23]. We recall the definition of Harish-Chandra type groups [23].

Let G be a Zariski connected $\mathbb{R}$-group with Lie algebra $\mathfrak{g}$ and let $G_{\mathbb{C}}$ be the complexification of G. Suppose there are given a Zariski connected $\mathbb{R}$-subgroup K of G with Lie algebra $\mathfrak{k}$ and connected unipotent $\mathbb{C}$ -subgroups $P_\pm$ of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{p}_\pm$. The group G is called *of Harish-Chandra type* if the following conditions are satisfied:

(HC 1) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-$ is a direct sum of vector spaces, $\left[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_\pm\right] \subset \mathfrak{p}_\pm$, and $\overline{\mathfrak{p}_+} = \mathfrak{p}_-$; (HC 2) the map $P_+ \times K_{\mathbb{C}} \times P_- \to G_{\mathbb{C}}$ gives a holomorphic injection of $P_+ \times K_{\mathbb{C}} \times P_-$ onto its open image $P_+K_{\mathbb{C}}P_-$; (HC 3) $G \subset P_+K_{\mathbb{C}}P_-$ and $G \cap K_{\mathbb{C}}P_- = K$.

If $g \in P_+K_{\mathbb{C}}P_- \subset G_{\mathbb{C}}$, we denote by $(g)_+ \in P_+$, $(g)_0 \in K_{\mathbb{C}}$, $(g)_- \in P_-$ the components of g such that $g = (g)_+(g)_0(g)_-$.

The identity connected component of a linear algebraic group H is denoted in the usual topology by $\mathring{H}$. The generalized Harish-Chandra embedding of the homogeneous space $\mathcal{D} = \mathring{G}/\mathring{K}$ into $\mathfrak{p}_+$ is defined by $g\mathring{K} \longmapsto z$, where $g \in \mathring{G}$, $z \in \mathfrak{p}_+$ and $\exp z = (g)_+$. Then the $\mathring{G}$-invariant complex structure of $\mathcal{D}$ is determined by the natural inclusion $\mathcal{D} \hookrightarrow P_+ \subset G_{\mathbb{C}}/(K_{\mathbb{C}}P_-)$. Let $(G_{\mathbb{C}} \times \mathfrak{p}_+)'$ be the set of elements $(g, z) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ such that $g \exp z \in P_+K_{\mathbb{C}}P_-$ and let $(\mathfrak{p}_+ \times \mathfrak{p}_+)'$ be the set of elements $(z_1, z_2) \in \mathfrak{p}_+ \times \mathfrak{p}_+$ such that $(\exp \bar{z}_2)^{-1} \exp z_1 \in P_+K_{\mathbb{C}}P_-$.

The *canonical automorphy factor* $J : (G_{\mathbb{C}} \times \mathfrak{p}_+)' \to K_{\mathbb{C}}$ and the *canonical kernel function* $K : (\mathfrak{p}_+ \times \mathfrak{p}_+)' \to K_{\mathbb{C}}$ for G are defined by

$$J(g, z) = (g \exp z)_0\,, \quad K(z', z) = (((\exp \overline{z})^{-1} \exp z')_0)^{-1}, \tag{2.3}$$

where $(g, z) \in (G_{\mathbb{C}} \times \mathfrak{p}_+)'$ and $(z', z) \in (\mathfrak{p}_+ \times \mathfrak{p}_+)'$.

According to [27], [13] (Corollary 4.5, Proposition 4.7, and equation (6.1)), we obtain

Theorem 2.1 *a) The Jacobi group G^J acts transitively on $\mathfrak{D}^J$ by*

$$gx = (\sigma w, v_{\sigma w} + {}^t(c\tau(w) + d)^{-1}z),\ \rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4}$$

where $g = (\sigma, v, \varkappa) \in G^J$ *and* $x = (w, z) \in \mathfrak{D}^J$.

b) The canonical automorphy factor J *for the Jacobi group* G^J *is given by*

$$J(g, x) = (J_1(\sigma, w), 0, J_2(g, x)), \tag{2.5}$$

where $g = (\sigma, v, \varkappa) \in G^J$, $x = (w, z) \in \mathfrak{D}^J$, J_1 *is the canonical automorphy factor for* G^s, *and*

$$J_2(g, x) = \varkappa + \frac{1}{2}D(v, v_{\sigma w}) + \frac{1}{2}D(2v + \rho(\sigma)z, J_1(\sigma, w)z). \tag{2.6}$$

c) The canonical kernel function K *for the Jacobi group* G^J *is given by*

$$K(x, x') = (K_1(w, w'), 0, K_2(x, x')), \tag{2.7}$$

where $x = (w, z) \in \mathfrak{D}^J$, $x' = (w', z') \in \mathfrak{D}^J$, K_1 *is the canonical kernel function for* G^s, *and*

$$K_2(x, x') = D(2\bar{z}' + \frac{1}{2}\overline{\tau(w')}z, qz) + \frac{1}{2}D(\bar{z}', q\tau(w)\bar{z}'),\ q = \rho(K_1(w, w'))^{-1}. \tag{2.8}$$

The Heisenberg group $\mathrm{H}_n(\mathbb{R})$ consists of all elements (λ, μ, κ), where $\lambda, \mu \in M_{1\,n}(\mathbb{R})$, $\kappa \in \mathbb{R}$ with the multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'). \tag{2.9}$$

Let $G^J_n = \mathrm{Sp}(n, \mathbb{R}) \ltimes \mathrm{H}_n(\mathbb{R})$ be the semidirect product of the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ and the Heisenberg group $\mathrm{H}_n(\mathbb{R})$ endowed with the following multiplication law:

$$(\sigma, (\lambda, \mu, \kappa)) \cdot (\sigma', (\lambda', \mu', \kappa')) = (\sigma\sigma', (\lambda\sigma', \mu\sigma', \kappa) \circ (\lambda', \mu', \kappa')), \tag{2.10}$$

where $(\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in \mathrm{H}_n(\mathbb{R})$ and $\sigma, \sigma' \in \mathrm{Sp}(n, \mathbb{R})$. The *Jacobi group* G^J_n *of degree* n acts transitively on the *Jacobi-Siegel space* $\mathfrak{H}^J_n = \mathfrak{H}_n \times \mathbb{C}^n$ by $g(\Omega, \zeta) = (\Omega_g, \zeta_g)$, where $(\Omega, \zeta) \in \mathfrak{H}^J_n$, $g = (\sigma, (\lambda, \mu, \kappa)) \in G^J_n$, σ is given by (2.1), and [29]

$$\Omega_g = (a\Omega + b)(c\Omega + d)^{-1},\ \zeta_g = \nu(c\Omega + d)^{-1},\ \nu = \zeta + \lambda\Omega + \mu. \tag{2.11}$$

According with Theorem 2.1 and [22], we have

Proposition 2.1 *The canonical automorphy factor* J_1 *and the canonical kernel function* K_1 *for* $\mathrm{Sp}(n, \mathbb{R})$ *are given by*

$$J_1(\sigma, \Omega) = \begin{pmatrix} {}^t(c\Omega + d)^{-1} & 0 \\ 0 & c\Omega + d \end{pmatrix}, \tag{2.12}$$

$$K_1(\Omega', \Omega) = \begin{pmatrix} 0 & \overline{\Omega} - \Omega' \\ (\Omega' - \overline{\Omega})^{-1} & 0 \end{pmatrix}, \tag{2.13}$$

where $\Omega, \Omega' \in \mathfrak{H}_n$ *and* $\sigma \in \mathrm{Sp}(n, \mathbb{R})$ *is given by* (2.1).

The canonical automorphy factor $\theta = J_2(g,(\Omega,\zeta))$ *for* G^J_n *is given by*

$$\theta = \kappa + \lambda\, {}^t\zeta + \nu\, {}^t\lambda - \nu(c\Omega + d)^{-1}c\, {}^t\nu, \quad \nu = \zeta + \lambda\Omega + \mu, \tag{2.14}$$

where $g = (\sigma, (\lambda,\, \mu, \kappa)) \in G^J_n$*,* σ *is given by* (2.1)*, and* $(\Omega,\zeta) \in \mathfrak{H}^J_n$*.*

The canonical automorphy kernel K_2 for G^J_n *is given by*

$$K_2\left((\zeta',\Omega'),(\zeta,\Omega)\right) = -\frac{1}{2}(\zeta' - \bar{\zeta})(\Omega' - \bar{\Omega}')^{-1}({}^t\zeta' - {}^t\bar{\zeta}), \tag{2.15}$$

where $(\Omega,\zeta),(\Omega',\zeta') \in \mathfrak{H}^J_n$*.*

Let $\mathfrak{D}_n$ *be the Siegel disk of degree* n consisting of all symmetric matrices $W \in M_n(\mathbb{C})$ with $I_n - W\bar{W} > 0$. Let $\mathrm{Sp}(n,\mathbb{R})_*$ be the multiplicative group of all matrices $\omega \in M_{2n}(\mathbb{C})$ such that

$$\omega = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad {}^tp\bar{p} - {}^t\bar{q}q = I_n, \; {}^tp\bar{q} = {}^t\bar{q}p, \quad p,q \in M_n(\mathbb{C}). \tag{2.16}$$

Remark that $\mathrm{Sp}(n,\mathbb{R})_* = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$ for $n > 1$. $\mathrm{Sp}(n,\mathbb{R})_*$ acts transitively on $\mathfrak{D}_n$ by $\omega W = (pW + q)(\bar{q}W + \bar{p})^{-1}$, where $\omega \in \mathrm{Sp}(n,\mathbb{R})_*$ and $W \in \mathfrak{D}_n$. Let $K_{n*} \cong \mathrm{U}(n)$ be the maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})_*$ consisting of all $\omega \in \mathrm{Sp}(n,\mathbb{R})_*$ given by (2.16) with $p \in \mathrm{U}(n)$ and $q = 0$. Then $\mathfrak{D}_n \cong \mathrm{Sp}(n,\mathbb{R})_*/\mathrm{U}(n)$.

Let G^J_{n*} be the Jacobi group consisting of all elements $(\omega,(\alpha,\varkappa))$, where $\omega \in \mathrm{Sp}(n,\mathbb{R})_*$, $\alpha \in \mathbb{C}^n$, $\varkappa \in \mathrm{i}\mathbb{R}$, and endowed with the multiplication law

$$(\omega',(\alpha',\varkappa'))(\omega,(\alpha,\varkappa)) = \left(\omega'\omega, \varkappa + \varkappa' + \beta^t\bar{\alpha} - \bar{\beta}^t\alpha\right), \tag{2.17}$$

where $(\omega,(\alpha,\varkappa)),(\omega',(\alpha',\varkappa')) \in G^J_{n*}$ $\beta = \alpha' p + \bar{\alpha}'\bar{q}$, and ω is given by (2.16).

The Heisenberg group $\mathrm{H}_n(\mathbb{R})_*$ consists of all elements $(I_n,(\alpha,\varkappa)) \in G^J_{n*}$, where $\omega \in \mathrm{Sp}(n,\mathbb{R})_*$, $\alpha \in \mathbb{C}^n$, $\varkappa \in \mathrm{i}\mathbb{R}$. The center $\mathcal{A}_* \cong \mathbb{R}$ of $\mathrm{H}_n(\mathbb{R})_*$ consists of all elements $(I_n,(0,\varkappa)) \in G^J_{n*}$ with $\varkappa \in \mathrm{i}\mathbb{R}$. According to [30], there exists an isomorphism $\Theta : G^J_n \to G^J_{n*}$ given by $\Theta(g) = g_*,\ g = (\sigma,(\lambda,\mu,\kappa)) \in G^J_n$, $g_* = (\omega,(\alpha,\varkappa)) \in G^J_{n*}$,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; \omega = \begin{pmatrix} p_+ & p_- \\ \overline{p}_- & \overline{p}_+ \end{pmatrix}, \tag{2.18}$$

$$p_\pm = \frac{1}{2}(a \pm d) \pm \frac{\mathrm{i}}{2}(b \mp c), \quad \alpha = \frac{1}{2}(\lambda + \mathrm{i}\mu), \quad \varkappa = -\mathrm{i}\frac{\kappa}{2}. \tag{2.19}$$

Let $\mathfrak{D}^J_n = \mathfrak{D}_n \times \mathbb{C}^n \cong G^J_{n*}/(\mathrm{U}(n) \times \mathbb{R})$ be *the Siegel-Jacobi disk of degree* n. G^J_{n*} acts transitively on $\mathfrak{D}^J_n$ by $g_*(W,z) = (W_{g_*}, z_{g_*})$, where $g_* = (\omega,(\alpha,\varkappa)) \in G^J_{n*}$, $(W,z) \in \mathfrak{D}^J_n$, ω is given by (2.18), and [30]

$$W_{g_*} = (pW + q)(\bar{q}W + \bar{p})^{-1}, \; z_{g_*} = (z + \alpha W + \bar{\alpha})(\bar{q}W + \bar{p})^{-1}. \tag{2.20}$$

We now consider a partial Cayley transform of the Siegel-Jacobi disk $\mathfrak{D}^J_n$ onto the Siegel-Jacobi space $\mathfrak{H}^J_n$ which gives a partially bounded realization of $\mathfrak{H}^J_n$ [30]. The *partial Cayley transform* $\phi : \mathfrak{D}^J_n \rightarrow \mathfrak{H}^J_n$ is defined by

$$\Omega = \mathrm{i}(I_n + W)(I_n - W)^{-1}, \quad \zeta = 2\,\mathrm{i}\,z(I_n - W)^{-1}, \tag{2.21}$$

where $(\zeta, \Omega) = \phi\left((W, z)\right)$ and $(W, z) \in \mathfrak{D}^J_n$.

ϕ is a a biholomorphic map which satisfies $g\phi = \phi g_*$ for any $g \in G^J_n$ and $g_* = \Theta(g)$ [30].

The inverse partial Cayley transform $\phi^{-1} : \mathfrak{H}^J_n \rightarrow \mathfrak{D}^J_n$ is given by

$$W = (\Omega - \mathrm{i}I_n)(\Omega + \mathrm{i}I_n)^{-1}, \quad z = \zeta(\Omega + \mathrm{i}I_n)^{-1}, \tag{2.22}$$

where $(W, z) = \phi^{-1}\left((\Omega, \zeta)\right) \in \mathfrak{D}^J_n$ and $(\Omega, \zeta) \in \mathfrak{H}^J_n$.

According with Theorem 2.1, [22] and [30], we have

Proposition 2.2 *The canonical automorphy factor J_{1*} and the canonical kernel function K_{1*} for $\mathrm{Sp}(n, \mathbb{R})_*$ are given by*

$$J_{1*}(\omega, W) = \begin{pmatrix} {}^t(\bar{q}W + \bar{p})^{-1} & 0 \\ 0 & \bar{q}W + \bar{p} \end{pmatrix}, \tag{2.23}$$

$$K_{1*}(W', W) = \begin{pmatrix} I_n - W'\bar{W} & 0 \\ 0 & {}^t(I_n - W'\bar{W})^{-1} \end{pmatrix}, \tag{2.24}$$

where $W, W' \in \mathfrak{D}_n$ and $\omega \in \mathrm{Sp}(n, \mathbb{R})_*$ is given by (2.16).

The canonical automorphy factor $\theta_* = J_2(g_*, (W, z))$ for G^J_* is given by

$$\theta_* = \kappa_* + z\,{}^t\alpha + \nu_*\,{}^t\alpha - \nu_*(\bar{q}W + \bar{p})^{-1}\bar{q}\,{}^t\nu_*, \quad \nu_* = z + \alpha W + \bar{\alpha}, \tag{2.25}$$

where $g_* = (\omega, (\alpha, \varkappa)) \in G^J_{n*}$, ω is given by (2.16)*, and $(W, z) \in \mathfrak{D}^J_n$.*

The canonical automorphy kernel for G^J_* is given by $K_{2*}((W', z'), (W, z)) = A(W', z'; W, z)$, where $(W, z), (W', z') \in \mathfrak{D}^J_n$, and

$$A(W', z'; W, z) = (\bar{z} + \frac{1}{2}z'\bar{W})(I_n - W'\bar{W})^{-1}\,{}^tz' + \frac{1}{2}\bar{z}(I_n - W'\bar{W})^{-1}W'\,{}^t\bar{z}. \tag{2.26}$$

3 Fock spaces based on the Siegel disk

Let $\mathcal{A}_* \cong \mathbb{R}$ be the center of the Heisenberg group $\mathrm{H}_{n*}(\mathbb{R})$. Given $m \in \mathbb{R}$, let χ^m be the central character of $\mathcal{A}_*$ defined by $\chi^m(\kappa) = \exp(2\pi\mathrm{i}m\kappa)$, $\kappa \in \mathcal{A}_*$. Suppose $m > 0$.

For each $W \in \mathfrak{D}_n$ we denote by $\mathcal{F}_{mW}$ the Fock space of all functions $\Phi \in \mathcal{O}(\mathbb{C}^n)$ such that $\|\Phi\|_{mW} < \infty$ and the inner product is defined by [22]

$$\begin{aligned} (\Phi, \Psi)_{mW} &= (2\pi m)^n \left(\det(1 - W\bar{W})\right)^{-1/2} \\ &\quad \times \int_{\mathbb{C}^n} \Phi(z)\overline{\Psi(z)} \exp(-8\pi m A(W, z)) d\nu(z), \end{aligned} \tag{3.1}$$

where the Lebesgue measure on $\mathbb{C}^n$ is given by

$$d\nu(\zeta) = \pi^{-n}\prod_{i=1}^{n} d\mathrm{Re}\zeta_i\, d\mathrm{Im}\zeta_i, \tag{3.2}$$

and $A(W,z) = K_{2*}((W,z),(W,z))$ can be written as

$$A(W,z) = (\bar{z} + \frac{1}{2}z\bar{W})(I_n - W\bar{W})^{-1}\,{}^t z + \frac{1}{2}\bar{z}(I_n - W\bar{W})^{-1}W\,{}^t\bar{z}. \tag{3.3}$$

Remark that $\mathcal{F}_{m\,0}$ is the usual Bargmann space [1].

We consider the Gaussian functions $G_U : \mathfrak{D}^J_n \to \mathbb{C}$, $U \in \mathbb{C}^n$, defined by $G_U(W,Z) = G(U,Z,W)$, where

$$G(U,Z,W) = \exp(U\,{}^tZ + \frac{1}{2}UW\,{}^tU) = \sum_{s\in\mathbb{N}^n}\frac{U^s}{s!}P_s(Z,W) \tag{3.4}$$

for all $(W,Z) \in \mathcal{D}^J_n$. We utilize the notation

$$U^s = \prod_{i=1}^{n} U_i^{s_i}, \quad s! = \prod_{i=1}^{n} s_i!, \quad |s| = \sum_{i=1}^{n} s_i, \quad \delta_{sr} = \prod_{i=1}^{n}\delta_{s_ir_i}, \tag{3.5}$$

where $U = (U_1,...,U_n) \in M_{1n}(\mathbb{C}) \cong \mathbb{C}^n$, $s = (s_1,...,s_n) \in \mathbb{N}^n$, and $r = (r_1,...,r_n) \in \mathbb{N}^n$. The polynomials $P_s : \mathcal{D}^J_n \to \mathbb{C}$, $s \in \mathbb{N}^n$, are exactly the matching functions studied by Neretin [17]. We express the homogeneous polynomial P_s of degree $|s|$ in the following compact form:

$$P_s(Z,W) = \sum_{a\in A_n,\,\tilde{a}\le s}\frac{s!}{2^{\hat{a}}a!(s-\tilde{a})!}Z^{s-\tilde{a}}W^a, \tag{3.6}$$

where A_n is the set of all symmetric matrices $a = (a_{ij})_{1\le i,\,j\le n}$ with $a_{ij} \in \mathbb{N}$,

$$W^a = \prod_{1\le i\le j\le n} W_{ij}^{a_{ij}}, \quad a! = \prod_{1\le i\le j\le n} a_{ij}, \quad \tilde{a}_k = \sum_{i=1}^{n} a_{ik}, \quad \hat{a} = \sum_{i=1}^{n} a_{ii}, \tag{3.7}$$

and $\tilde{a} \le s$ is equivalent with $\tilde{a}_i \le s_i$ for $1 \le i \le n$. Using the equations

$$\int_{\mathbb{C}^n} U^s\bar{U}^r d\nu(U) = \delta_{sr}s!, \tag{3.8}$$

$$\int_{\mathbb{C}^n} G(U,Z',W')G(\bar{U},\bar{Z},\bar{W})d\nu(U) = \det(1-W'\bar{W})^{-1/2}\exp A(W',z';W,z), \tag{3.9}$$

where $A(W',z';W,z)$ is defined by (2.26), we obtain

$$\left(\det(1-W'\bar{W})\right)^{-1/2}\exp A(W',z';W,z) = \sum_{s\in\mathbb{N}^n}\frac{1}{s!}P_s(Z',W')\overline{P_s(Z,W)}. \tag{3.10}$$

Equation (3.9) is given in [11] (Lemma 5).

We now define the polynomials $\Phi_{Ws} \in \mathcal{F}_{mW}$, $s \in \mathbb{N}^n$, by

$$\Phi_{Ws}(z) = \frac{1}{\sqrt{s!}}P_s(2\sqrt{2\pi m}z, W), \; s \in \mathbb{N}^n. \tag{3.11}$$

Proposition 3.1 *Given $W \in \mathfrak{D}_n$, the set of polynomials $\{\Phi_{Ws}|s \in \mathbb{N}^n\}$ forms an orthonormal basis of the Fock space $\mathfrak{F}_{mW}$. The kernel function of $\mathfrak{F}_{mW}$ admits the expansion*

$$\left(\det(1 - W\bar{W})\right)^{-1/2} \exp\left(2\pi m A(W, z'; W, z)\right) = \sum_{s \in \mathbb{N}^n} \Phi_{Ws}(z')\overline{\Phi_{Ws}(z)}. \quad (3.12)$$

Proof. Given $U \in \mathbb{C}^n$ and $W \in \mathcal{D}_n$, we define the function $\Psi_{UW} : \mathbb{C}^n \to \mathbb{C}$ such that $\Psi_{UW}(z) = G(U, 2\sqrt{2\pi m}z, W)$. Using the change of variables $Z = 2\sqrt{2\pi m}z$, we have

$$\|\Psi_{UW}\|^2_{mW} = \pi^{-n}\det(1-W\bar{W})^{-1/2}\int_{\mathbb{C}^n} \exp(B(U, Z, W) - A(Z, W))d\nu(Z)\,, \quad (3.13)$$

where

$$B(U, Z, W) = U^t Z + \bar{U}Z^\dagger - \frac{1}{2}UW\,{}^tU - \frac{1}{2}\bar{U}\bar{W}U^+. \quad (3.14)$$

Using the change of variables $Z' = (1 - W\bar{W})^{-1/2}\left(Z - \bar{U} - WU\right)$, the relation $d\nu(Z) = \det(1 - W\bar{W})d\nu(Z')$, and the relation [1]

$$\int_{\mathbb{C}^n} \exp\left(-\overline{Z'}^t Z' - \frac{1}{2}(Z'\bar{W}^t Z' + \overline{Z}' W Z'^\dagger)\right)d\nu(Z') = \pi^n \left(\det(1-W\bar{W})\right)^{-1/2}, \quad (3.15)$$

we obtain $\|\Psi_{UW}\|^2_{mW} = \exp(UU^\dagger)$. Then

$$\sum_{s,r \in \mathbb{N}^n} \frac{U^s\bar{U}^r}{\sqrt{s!r!}}(\Phi_{Ws}, \Phi_{Wr})_{mW} = \sum_{s \in \mathbb{N}^n} \frac{1}{s!}U^s\bar{U}^s. \quad (3.16)$$

By comparing the coefficients of $U^s\bar{U}^r$ in the series of both sides of (3.16), we see that

$$(\Phi_{Ws}, \Phi_{Wr})_{mW} = \delta_{sr}s!, \quad s, r \in \mathbb{N}^n. \quad (3.17)$$

Using (3.10) and (3.11), we obtain the expansion (3.12). ■

We now introduce the set of polynomials $f_s : \mathcal{D}^J_n \to \mathbb{C}$, $s \in \mathbb{N}^n$, defined by

$$f_s(W, z) = \frac{1}{\sqrt{s!}}P_s(2\sqrt{2\pi m}z, W). \quad (3.18)$$

Let $\mathcal{H}_0(\mathcal{D}^J_n)$ be the complex linear subspace of all holomorphic functions $f \in \mathcal{O}(\mathcal{D}^J_n)$ with the basis $\{f_s|s \in \mathbb{N}^n\}$. Let $\mathfrak{F}_m(\mathcal{D}^J_n)$ be the Hilbert space of all functions $f \in \mathcal{O}(\mathcal{D}^J_n)$ such that $\langle f, f\rangle_m < \infty$, where the inner product $\langle ., .\rangle_m$ is defined such that the set $\{f_s|s \in \mathbb{N}^n\}$ is an orthonormal basis. We now prove

Proposition 3.2 *a) The generating function of the basis $\{f_s|s \in \mathbb{N}^n\}$ can be expressed as*

$$\exp(8\pi m U\,{}^tz + \frac{1}{2}UW\,{}^tU) = \sum_{s \in \mathbb{N}^n} \frac{U^s}{\sqrt{s!}}f_s(W, z). \quad (3.19)$$

The kernel function of $\mathfrak{F}_m(\mathcal{D}_n^J)$ admits the expansion

$$\left(\det(1-W'\bar{W})\right)^{-1/2}\exp A(W',z';W,z)=\sum_{s\in\mathbb{N}^n}f_s(W',z')\overline{f_s(W,z)}. \quad (3.20)$$

b) $f\in\mathcal{O}(\mathcal{D}_n^J)$ is a solution of the system of differential equations

$$\frac{\partial^2 f}{\partial z_j\partial z_k}=8\pi m(1+\delta_{jk})\frac{\partial f}{\partial W_{jk}},\ 1\leq j\leq k\leq n, \quad (3.21)$$

if and only if $f\in\mathcal{H}_0(\mathcal{D}_n^J)$.

Proof. Using (3.4) and (3.18), we obtain (3.19). The generating function (3.19) satisfies (3.21). Then

$$\frac{\partial^2 f_s}{\partial z_j\partial z_k}=8\pi m(1+\delta_{jk})\frac{\partial f_s}{\partial W_{jk}},\ 1\leq j\leq k\leq n,\ s\in\mathbb{N}^n. \quad (3.22)$$

Using (3.6) and (3.18), we obtain

$$f_s(z,W)=\frac{1}{\sqrt{s!}}(2\sqrt{2\pi m}z)^s+R_s(z,W), \quad (3.23)$$

where R_s is a polynomial of degree $|s|-1$ in z. Then there exists the change of basis $\{z^sW^a|s\in\mathbb{N},a\in A_n\}\longmapsto\{f_s(z,W)W^a|s\in\mathbb{N}^n,a\in A_n\}$ in $\mathcal{O}(\mathcal{D}_n^J)$. Let $f\in\mathcal{O}(\mathcal{D}_n^J)$. Then there exists the set $\{c_s|c_s\in\mathcal{O}(\mathcal{D}_n),\ s\in\mathbb{N}^n\}$ such that

$$f(z,W)=\sum_{s\in\mathbb{N}^n}c_s(W)f_s(W,z). \quad (3.24)$$

If f satisfies (3.21), then $\partial c_s/\partial W_{jk}=0$ for any $1\leq j\leq k\leq n,\ s\in\mathbb{N}^n$. Then c_s is constant for any $s\in\mathbb{N}^n$. Hence $f\in\mathcal{O}(\mathcal{D}_n^J)$. The inverse implication follows from (3.22). ■

In the case $n=1$, Proposition 3.2 has been obtained in [6].

4 Scalar holomorphic discrete series of the Jacobi group on the Siegel-Jacobi disk

Consider the Jacobi group G_n^J. Let δ be a rational representation of $GL(n,\mathbb{C})$ such that $\delta|_{\mathrm{U}(n)}$ is a scalar irreducible representation of the unitary group $\mathrm{U}(n)$ with highest weight k, $k\in\mathbb{Z}$, and $\delta(A)=(\det A)^k$ [31]. Let $m\in\mathbb{R}$. Let $\chi=\delta\otimes\bar{\chi}^m$, where the central character χ^m of $\mathcal{A}\cong\mathbb{R}$ is defined by $\chi^m(\kappa)=\exp(2\pi i m\kappa)$, $\kappa\in\mathcal{A}$. Any scalar holomorphic irreducible representation of G_n^J is characterized by an index m and a weight k. Suppose $m>0$ and $k>n+1/2$.

Let $\mathcal{H}^{mk}$ denote the Hilbert space of all holomorphic functions $\varphi\in\mathcal{O}(\mathfrak{H}_n^J)$ such that $\|\varphi\|_{\mathfrak{H}_n^J}<\infty$ with the inner product defined by [25]

$$(\varphi,\psi)_{\mathfrak{H}^J_n} = C\int_{\mathfrak{H}^J_n} \varphi(\Omega,\zeta)\overline{\psi(\Omega,\zeta)}\,\mathcal{K}^{mk}(\Omega,\zeta)^{-1}d\mu(\Omega,\zeta), \tag{4.1}$$

where C is a positive constant, $(\Omega,\zeta)\in\mathfrak{H}^J_n$ and the G^J_n-invariant measure on $\mathfrak{H}^J_n$ is given by

$$d\mu(\Omega,\zeta) = (\det Y)^{-n-2}\prod_{1\le i\le n} d\xi_i\, d\eta_i \prod_{1\le j\le k\le n} dX_{jk}\, dY_{jk}. \tag{4.2}$$

Here $\xi=\mathrm{Re}\,\zeta,\ \eta=\mathrm{Im}\,\zeta,\ X=\mathrm{Re}\,\Omega,\ Y=\mathrm{Im}\,\Omega$.

The kernel function $\mathcal{K}^{mk}$ is defined by [25]

$$\mathcal{K}^{mk}(\Omega,\zeta) = \mathsf{K}^{mk}((\Omega,\zeta),(\Omega,\zeta)) = \exp\left(4\pi m\eta Y^{-1}\,{}^t\eta\right)(\det Y)^k, \tag{4.3}$$

$$\mathsf{K}^{mk}((\zeta',\Omega'),(\zeta,\Omega)) = \left(\det(\frac{\mathrm{i}}{2}\bar{\Omega}-\frac{\mathrm{i}}{2}\Omega')\right)^{-k}\exp(2\pi\mathrm{i}mK\left((\zeta',\Omega'),(\zeta,\Omega)\right)), \tag{4.4}$$

where K is given by (2.15).

Let π^{mk} be the unitary representation of G^J_n on $\mathcal{H}^{mk}$ defined by [25]

$$\left(\pi^{mk}(g^{-1})\varphi\right)(\Omega,\zeta) = \mathcal{J}^{mk}(g,(\Omega,\zeta))\varphi(\Omega_g,\zeta_g), \tag{4.5}$$

where $\varphi\in\mathcal{H}^{mk}$, $g\in G^J_n$, $(\Omega,\zeta)\in\mathfrak{H}^J_n$ and $(\Omega_g,\zeta_g)\in\mathfrak{H}^J_n$ is given by (2.11).

The automorphic factor $\mathcal{J}^{mk}$ for G^J_n is defined by [25]

$$\mathcal{J}^{mk}(g,(\zeta,\Omega)) = (\det(c\Omega+d))^{-k}\exp(2\pi\mathrm{i}m\theta), \tag{4.6}$$

where θ is given by (2.14) and σ is given by (2.1).

Takase proved the following theorem [25], [26]:

Theorem 4.1 *Suppose $k>n+1/2$. Then $\mathcal{H}^{mk}\neq\{0\}$ and π^{mk} is an irreducible unitary representation of G^J_n which is square integrable modulo center.*

Let $\mathcal{H}^{mk}_*$ denote the complex pre-Hilbert space of all $\psi\in\mathcal{O}(\mathfrak{D}^J_n)$ such that $\|\psi\|_{\mathfrak{D}^J_n}<\infty$ with the inner product defined by

$$(\psi_1,\psi_2)_{\mathfrak{D}^J_n} = C_*\int_{\mathfrak{D}^J_n}\psi_1(W,z)\,\overline{\psi_2(W,z)}\left(\mathcal{K}^{mk}_*(W,z)\right)^{-1}d\,\nu(W,z), \tag{4.7}$$

where C_* is a positive constant, $(z,W)\in\mathfrak{D}^J_n$,

$$\mathcal{K}^{mk}_*(W,z) = \left(\det(I_n-W\bar{W})\right)^{-k}\exp(8\pi mA(W,z)), \tag{4.8}$$

where A is given by (3.3) and the G^J_n-invariant measure on $\mathfrak{D}^J_n$ is [30]

$$d\nu(W,z) = (\det(1-W\bar{W}))^{-n-2}\prod_{i=1}^{n} d\,\mathrm{Re}z_i\, d\,\mathrm{Im}z_i\prod_{1\le j\le k\le n} d\,\mathrm{Re}\,W_{jk}\, d\,\mathrm{Im}W_{jk}. \tag{4.9}$$

According with [21], [30], and (2.26), the kernel function $\mathcal{K}^{mk}_*$ is given by $\mathcal{K}^{mk}_*(W,z)=\mathsf{K}^{mk}_*((W,z),(W,z))$, where

$$\mathsf{K}^{mk}_*((z,W),(z',W')) = \left(\det(I_n-W'\bar{W})\right)^{-k}\exp\left(8\pi mA(W',z';W,z)\right). \tag{4.10}$$

Remark 4.1 Using the coherent state method, Kramer, Saraceno, and Berceanu investigated the kernel (4.8) in the case $8\pi m = 1$ [12], [2]-[6].

We now introduce the map $g_* \longmapsto \pi_*^{mk}(g_*)$, where $\pi_*^{mk}(g_*)$: $\mathcal{H}_*^{mk} \to \mathcal{H}_*^{mk}$ is defined by

$$\left(\pi_*^{mk}(g_*^{-1})\psi\right)(z, W) = J_*^{mk}(g_*, (z, W))\psi(z_{g_*}, W_{g_*}), \tag{4.11}$$

$\psi \in \mathcal{H}_*^{mk}$, $g_* = (\omega, (\alpha, \varkappa)) \in G_{n*}^J$, $(z, W) \in \mathfrak{D}_n^J$, and $(z_{g_*}, W_{g_*}) \in \mathfrak{D}_n^J$ is given by (2.20). The automorphic factor J_*^{mk} for G_{n*}^J is defined by [21], [30]

$$J_*^{mk}(g_*, (z, W)) = \exp(2\pi \mathrm{i} m\theta_*) \left(\det(\bar{q}W + \bar{p})\right)^{-k} \tag{4.12}$$

where θ_* is given by (2.25) and ω given by (2.16).

Proposition 4.1 *Suppose $m > 0$, $k > n + 1/2$, and $C = 2^{n(n+3)}C_*$. Then*

a) $\mathcal{H}_*^{mk} \neq \{0\}$ and π_*^{mk} is an irreducible unitary representation of G_{n*}^J on the Hilbert space $\mathcal{H}_*^{mk}$ which is square integrable modulo center.

b) There exists the unitary isomorphism $T_*^{mk} : \mathcal{H}_*^{mk} \to \mathcal{H}^{mk}$ given by

$$\varphi(\Omega, \zeta) = \psi\ (W, z) \left(\det(I_n - W)\right)^k \exp(4\pi m z(I_n - W)^{-1\,t} z), \tag{4.13}$$

where $\psi \in \mathcal{H}_*^{mk}$, $\varphi = T_*^{mk}(\psi)$, $(W, z) \in \mathfrak{D}_n^J$, $(\Omega, \zeta) = \phi((-W, z)) \in \mathfrak{H}_n^J$, and ϕ is given by (2.21).

The inverse isomorphism $T^{mk} : \mathcal{H}^{mk} \to \mathcal{H}_*^{mk}$ is given by

$$\psi\ (W, z) = \varphi(\Omega, \zeta)\ \left(\det(I_n - \mathrm{i}\Omega)\right)^k \exp\left(2\pi m\zeta(I_n - \mathrm{i}\Omega)^{-1\,t}\zeta\right), \tag{4.14}$$

where $\psi \in \mathcal{H}_*^{mk}$, $\psi = T^{mk}(\varphi)$, $(\Omega, \zeta) \in \mathfrak{H}_n^J$, $(-W, z) = \phi^{-1}((\Omega, \zeta)) \in \mathfrak{D}_n^J$, and ϕ^{-1} is given by (2.22).

c) The representations π^{mk} and π_*^{mk} are unitarily equivalent.

Proof. Using the partial Cayley transform (2.21) and (2.22), we obtain

$$Y = \operatorname{Im}\Omega = (I_n - W)^{-1}(I_n - W\bar{W})(I_n - \bar{W})^{-1}, \tag{4.15}$$

$$\eta = \operatorname{Im}\zeta = z(I_n - W)^{-1} + \bar{z}(I_n - \bar{W})^{-1}. \tag{4.16}$$

By (4.15) and (4.16), we obtain

$$\eta Y^{-1\,t}\eta = 2A(z, -W) - z(I_n - W)^{-1\,t}z - \bar{z}(I_n - \bar{W})^{-1\,t}\bar{z}, \tag{4.17}$$

where A is given by (3.3). Using (4.2), (4.9), (2.21), and (2.22), in the limit $\Omega \to \mathrm{i}I_n$ and $W \to 0$, we obtain

$$d\mu(\zeta, \Omega) = 2^{n(n+3)} d\nu(z, W). \tag{4.18}$$

By (4.1), (4.7), (4.13), (4.14), the condition $C = 2^{n(n+3)}C_*$, and the change of variables $W \to -W$, we get $\|\varphi\|_{\mathfrak{H}_n^J} = \|\psi\|_{\mathfrak{D}_n^J}$. From $\zeta(I_n - \mathrm{i}\Omega)^{-1\,t}\zeta = -2\,z(I_n - W)^{-1\,t}z$ it is clear that (4.13) and (4.14) are equivalent. By Theorem 4.1, a) and b) hold. Using (2.21), (2.22), (4.5), (4.11), (4.13), and (4.14), we obtain $\pi^{mk}T^{mk} = T_*^{mk}\pi_*^{mk}$. ■

Remark 4.2 Berndt, Böcherer and Schmidt constructed the holomorphic discrete series of the Jacobi group in the case $n = 1$ [8], [9].

Let $\mathcal{H}^k$ denote the complex Hilbert space of all holomorphic functions $\Phi \in \mathcal{O}(\mathfrak{D}_n)$ such that $\|\Phi\|_{\mathfrak{D}_n} < \infty$, with the inner product defined by

$$(\Psi_1, \Psi_2)_k = \int_{\mathcal{D}_n} \Psi_1(W)\overline{\Psi_2(W)} \left(\det(1 - W\bar{W})\right)^{k-1/2} d\mu_{\mathcal{D}_n}(W), \tag{4.19}$$

$$d\mu_{\mathcal{D}_n}(W) = \left(\det(1 - W\bar{W})\right)^{-n-1} \prod_{1\le j\le k\le n} \mathrm{d}\,\mathrm{Re}\, W_{jk}\, \mathrm{d}\,\mathrm{Im}\, W_{jk}.$$

We have $\mathcal{H}^k \neq \{0\}$ for $k > n+1/2$ [7], [25]. Let $\{Q_a | a \in A_n\}$ be an orthonormal polynomial basis of $\mathcal{H}^k$.

We introduce the polynomials

$$F_{sa}(W, z) = \sqrt{\frac{(8\pi m)^n}{C_* s!}} P_s(\sqrt{8\pi m} z, W) Q_a(W), \;\; s \in \mathbb{N}^n, \;\; a \in A_n. \tag{4.20}$$

Proposition 4.2 *The set of polynomials* $\{F_{sa} | s \in \mathbb{N}^n, \; a \in A_n\}$ *forms an orthonormal basis of* $\mathcal{H}^{mk}_*$. *The kernel function of* $\mathcal{H}^{mk}_*$ *satisfies the expansion*

$$(\det(1 - W'\bar{W}))^{-k} \exp A(W', z', W, z) = \sum_{s\in\mathbb{N}^n, a\in A_n} F_{sa}(W', z')\overline{F_{sa}(W, z)}. \tag{4.21}$$

Proof. We introduce the functions $F_U : \mathfrak{D}^J_n \to \mathbb{C}$, $U \in \mathbb{C}^n$, such that $F_U(W, z) = G(U, 2\sqrt{2\pi m} z, W)$. Using (4.20) and the proof of Proposition 3.1, we have

$$\begin{aligned}(F_U(W, z)Q_a, F_U(W, z)Q_b)_{\mathfrak{D}^J_n} &= C_* (8m)^n \exp(UU^\dagger) \\ &\times \int_{\mathcal{D}_n} Q_a(W)\overline{Q_b(W)} \det(1 - W\bar{W})^{k-1/2} d\mu_{\mathcal{D}_n}(W),\end{aligned} \tag{4.22}$$

$$(F_{sa}, F_{rb})_{\mathfrak{D}^J_n} = \delta_{sr}\delta_{ab}, \;\; s, r \in \mathbb{N}^n, \;\; a, b \in A_n. \tag{4.23}$$

The Berezin kernel of $\mathcal{H}^k$ is positive definite for $k > n + 1/2$ [7] and satisfies the following identity:

$$\left(\det(1 - W'\bar{W})\right)^{-k+1/2} = \sum_{a\in A_n} Q_a(W')\overline{Q_a(W)}. \tag{4.24}$$

Using (3.12) and (4.24), we obtain (4.21). ■

Remark 4.3 In the case $n = 1$ and $8\pi m = 1$, the expansion (4.21) was obtained in [3], using the coherent state method.

Remark 4.4 We now discuss the unitary representations of Jacobi groups based on Siegel-Jacobi domains in the language of coherent states [18]. Let $Q(\mathcal{H})$ be the set of all one-dimensional projections of the Hilbert space $\mathcal{H}$. Let $P[\psi]$ denote the one-dimensional projection determined by $\psi \in \mathcal{H}\backslash\{0\}$. The elements of $Q(\mathcal{H})$ can be considered either as normal pure states of the von Neumann algebra of bounded operators on $\mathcal{H}$ or as pure states of the C^*-algebra of compact operators on $\mathcal{H}$ [10]. The projective Hilbert space $P(\mathcal{H})$ consists of all one-dimensional complex linear subspaces of $\mathcal{H}$. The space $P(\mathcal{H})$ is a Kähler manifold equipped with the usual Fubini-Study metric [10]. The space $Q(\mathcal{H})$ with relative w^* -topology is homeomorphic to $P(\mathcal{H})$ with the manifold topology [10]. Then we can identify $Q(\mathcal{H})$ with $P(\mathcal{H})$.

We recall an intrinsic definition of coherent state representations given in [15].

Let G be a connected, simply connected Lie group and $\mathfrak{X}$ a G-homogeneous space which admits an invariant measure $\mu_{\mathfrak{X}}$. Let π be a continuous irreducible unitary representation of G in the separable Hilbert space $\mathcal{H}$. A family $\mathcal{E} = \{E_x|x \in \mathfrak{X}\}$ of one-dimensional projections in $\mathcal{H}$ will be called a π-*system of coherent states* based on $\mathfrak{X}$ if the following conditions are satisfied: 1) $E_{g\,x} = \pi\,(g)E_x\pi\,(g)^{-1}$ for any $g \in G$ and $x \in \mathfrak{X}$; 2) there exists $\psi \in \mathcal{H}\backslash\{0\}$, such that $\int_{\mathfrak{X}} |\langle\psi, \pi\,(g)\psi\rangle|^2\, d\,\mu_{\mathfrak{X}} < \infty$. π is called a *symplectic (Kähler) coherent state representation* if $\mathcal{E}$ and $\mathfrak{X}$ are isomorphic symplectic (Kähler) manifolds and $\mathfrak{X}$ is a symplectic (Kähler) submanifold of $Q(\mathcal{H})$.

Moscovici and Verona have been studied coherent state representations based precisely on the coadjoint orbit associated with π in the sense of geometric quantization [15]. The Schrödinger coherent state systems for the Heisenberg group with one-dimensional center on the Fock spaces of holomorphic functions have been obtained by Bargmann [1], Satake [20], [22], [23], and Lee [13].

Lisiecki and Neeb investigated some Kähler coherent state representations of Heisenberg groups and Jacobi groups with one-dimensional center [14],[16]. The orbit method for the Heisenberg group and the Jacobi group with multi-dimensional center has been studied in detail by Yang [28].

Let π be an irreducible unitary representation of the Jacobi group G^J with the Jacobi-Siegel domain $\mathfrak{D}$ and the kernel function $K : \mathfrak{D} \times \mathfrak{D} \to$Hom$(W, W)$. The representation space $\mathcal{H}$ consists of holomorphic functions taking their values in a finite dimensional Hilbert space W. For each $x \in \mathfrak{D}$ and $v \in W$, we consider the vectors $K_{xv} \in \mathcal{H}$ given by $K_{xv}(x') = K(x, x')v$ for any $x' \in \mathfrak{D}$. Then $\{P[K_{xv}]|x \in \mathfrak{D},\ v \in W\}$ is a π-system of coherent states. In particular, the π^{mk}-system of coherent states based on $\mathfrak{H}^J_n$ and the π^{mk}_*-system of coherent states based on $\mathfrak{D}^J_n$ are determined by the explicit kernel functions given by (4.4) and (4.10), respectively.

Acknowledgments. The authors are indebted to the Organizing Committee of *The International Conference of Differential Geometry and Dynamical Systems*, University Politehnica of Bucharest, Romania, for the opportunity to report results at the meeting. S. B. was partially supported by the CNCSIS-UEFISCSU project PNII-IDEI 454/2009, CNCSIS Cod ID-44, A.G. was partially supported by the CNCSIS-

UEFISCSU project PNII- IDEI 545/2009, CNCSIS Cod ID-1089 and both the authors have been partially supported by the ANCS project program PN 09 37 01 02/2009.

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