Description of the 2νν$\beta\beta$ decay within a gauge restored of a fully renormalized pnQRPA approach

2011 J. Phys. G: Nucl. Part. Phys. 38 055102

(http://iopscience.iop.org/0954-3899/38/5/055102)

J. Phys. G: Nucl. Part. Phys. **38** (2011) 055102 (18pp) doi:10.1088/0954-3899/38/5/055102

Description of the $2\nu\nu\beta\beta$ decay within a gauge restored of a fully renormalized pnQRPA approach

C M Raduta[1] **and A A Raduta**[1,2]

[1] Department of Theoretical Physics, Institute of Physics and Nuclear Engineering, PO Box MG6, Bucharest, Romania
[2] Academy of Romanian Scientists, 54 Splaiul Independentei, Bucharest 050094, Romania

E-mail: raduta@nipne.ro

Received 23 January 2011
Published 22 March 2011
Online at stacks.iop.org/JPhysG/38/055102

Abstract
A many-body Hamiltonian involving the mean field for a projected spherical single particle basis, the pairing interactions for alike nucleons, the dipole–dipole proton–neutron interactions in the particle–hole (ph) channel and the ph dipole-pairing potential is treated by the projected gauge of fully renormalized proton–neutron quasiparticle random phase approximation (PGFRpnQRPA) approach. The resulting wavefunctions and energies for the mother and the daughter nuclei are used to calculate the $2\nu\beta\beta$ decay rate and the process half life. For illustration, the formalism is applied for the decays ^{100}Mo→ ^{100}Ru and ^{116}Cd→ ^{116}Sn. The results are in good agreement with the corresponding experimental data. The Ikeda sum rule is obeyed.

1. Introduction

Double beta decay is one of the most exciting topics of nuclear physics because of the possible existence of neutrinoless $\beta\beta$ decay modes, which are extensively searched since they connect to the neutrino mass and neutrino nature. The calculation of the nuclear matrix elements for these decay modes is a challenge in the study of the $\beta\beta$ decay process; however, this calculation implies the calculation not only of Gamow–Teller transitions but also of Fermi and tensor transitions.

The $2\nu\beta\beta$ process is interesting by its own but is also very attractive because it constitutes a test for some of the nuclear matrix elements (m.e.) which are used for the process of $0\nu\beta\beta$ decay. The subject development is reflected by several review papers [1–7]. Our contribution described in this paper concerns the $2\nu\beta\beta$ process, which can be viewed as two consecutive and virtual single β^- decays. The formalism yielding closest results to the experimental data is the proton–neutron random phase approximation (pnQRPA) which includes the particle–hole (ph) and particle–particle (pp) [8] as independent two-body interactions. The second leg of the $2\nu\beta\beta$ process is very sensitive to changing the relative strength of the later interaction, denoted hereafter by g_{pp}. It is worth mentioning that the two-body interaction of ph type is repulsive while that of pp nature is attractive. Due to this feature there is a critical value

for g_{pp} for which the first root of the pnQRPA equation vanishes. Actually, this is the signal that the pnQRPA approach is no longer valid. Moreover, the g_{pp} value which corresponds to a transition amplitude which agrees with the corresponding experimental data is close to the mentioned critical value. That means that the result is not stable to adding corrections to the pnQRPA picture. One improvement for the pnQRPA was achieved by one of us (AAR), in collaboration, in [9, 10], by using a boson expansion (BE) procedure. Another procedure of going beyond pnQRPA is to renormalize the dipole two quasiparticle operators by replacing the scalar components of their commutators by their average values [11]. Such a renormalization is inconsistently achieved since the scattering operators are not renormalized. This lack of consistency was removed in [12, 13] where a fully renormalized pnQRPA (FRpnQRPA) is proposed.

Unfortunately, all higher pnQRPA procedures mentioned above have a common drawback of violating the Ikeda sum rule (ISR) by an amount of about 20–30% [14]. It is believed that such a violation is caused by the gauge symmetry breaking. Consequently, a method of restoring this symmetry was formulated by the present authors in [15].

In this paper the results of [15] are improved in three respects: (a) aiming at providing a unitary description of the process for the situations when the nuclei involved are spherical or deformed, here we use the projected spherical single particle basis defined in [16] and used for double beta decay in [17, 18]; (b) the space of proton–neutron dipole configurations is split into three subspaces, one being associated with the single β^-, one to the β^+ process, and one spanned by the unphysical states; (c) the correlations for the second leg of the process are mainly determined by the ph dipole-pairing term. A compact expression for the dispersion equation of energies is obtained from the linearized equations of motion of the basic transition operators corresponding to the two coupled processes. The numerical application is made for the $2\nu\beta\beta$ processes $^{100}Mo \rightarrow ^{100}$ Ru and ^{116}Cd$\rightarrow$ ^{116}Sn.

We shall describe the formalism and results according to the following plan. The single particle basis is briefly presented in section 2. The model Hamiltonian is given in section 3. The FRpnQRPA approach is discussed in section 4, while the projected gauge of FRpnQRPA (PGFRpnQRPA) is the objective of section 5. The Gamow–Teller (GT) amplitude for the $2\nu\beta\beta$ process is given in section 6. Numerical applications are shown in section 7, while the final conclusions are drawn in section 8.

2. The model Hamiltonian

We suppose that the states describing the mother, the daughter and the intermediate odd–odd nuclei are described by a sole many-body Hamiltonian. Written in the second quantization with a projected spherical single particle basis [16, 17, 19–23], this is similar to the one used in [18], H', with the difference that the proton–neutron dipole–dipole interaction in the particle–particle (pp) channel, H_{pp}, is replaced by an attractive term of dipole-pairing type, H_{dp}:

$$H = \sum \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha})c^{\dagger}_{\tau\alpha IM}c_{\tau\alpha IM} - \sum \frac{G_\tau}{4} P^{\dagger}_{\tau\alpha I} P_{\tau\alpha I'} + 2\chi \sum \beta^-_\mu(pn)\beta^+_{-\mu}(p'n')(-)^\mu - X_{dp} \sum_{\substack{pn;p'\\ n';\mu}} \big(\beta^-_\mu(pn)\beta^-_{-\mu}(p'n') + \beta^+_{-\mu}(p'n')\beta^+_\mu(pn)\big)(-1)^{1-\mu}. \tag{2.1}$$

The operator $c^{\dagger}_{\tau\alpha IM}$ ($c_{\tau\alpha IM}$) creates (annihilates) a particle of type τ ($=p, n$) in the state Φ^{IM}_{α}, when acting on the vacuum state $|0\rangle$. In order to simplify the notations, hereafter the set

of quantum numbers $\alpha(=nlj)$ will be omitted. The two-body interaction consists of three terms, the pairing, the dipole–dipole particle–hole (ph) and the dipole-pairing interaction. The strengths corresponding to the two-body interactions are denoted by G_τ ($\tau = p, n$), χ, X_{dp}, respectively. All of them are separable interactions, with the factors defined by the following expressions:

$$\begin{aligned} P^\dagger_{\tau I} &= \sum_M \frac{2}{2I+1} c^\dagger_{\tau IM} c^\dagger_{\widetilde{\tau IM}}, \\ \beta^-_\mu(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I}'} c^\dagger_{pIM} c_{nI'M'}. \end{aligned} \tag{2.2}$$

The remaining operators from equation (2.2) can be obtained from the above-defined operators, by Hermitian conjugation.

We note that the Hamiltonian used in this paper differs from that of [18] by the attractive two-body interaction. Indeed, while in the quoted reference a dipole–dipole pp interaction is used here a proton–neutron dipole-pairing term is introduced. As shown in the next section, the reason for such a replacement consists of the fact that fixing the gauge the pp interaction does not contribute at all at the pnQRPA level.

Passing to the quasiparticle (qp) representation through the Bogoliubov–Valatin transformation

$$a^\dagger_{\tau IM} = U_{\tau I} c^\dagger_{\tau IM} - s_{IM} V_{\tau I} c_{\tau I-M}, s_{IM} = (-)^{I-M}, \quad \tau = p, n, \quad U^2_{\tau I} + V^2_{\tau I} = 1, \tag{2.3}$$

the first two terms of H are replaced by the independent quasiparticles term, $\sum E_{\tau I} a^\dagger_{\tau IM} a_{\tau IM}$, while the two-body dipole–dipole interactions are expressed in terms of the dipole two qp and the dipole qp density operators:

$$\begin{aligned} A^\dagger_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^\dagger_{pI_pm_p} a^\dagger_{nI_nm_n}, \qquad A_{1\mu}(pn) = \left(A^\dagger_{1\mu}(pn)\right)^\dagger, \\ B^\dagger_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^\dagger_{pj_pm_p} a_{nI_nm_n} (-)^{I_n-m_n}, \qquad B_{1\mu}(pn) = \left(B^\dagger_{1\mu}(pn)\right)^\dagger. \end{aligned} \tag{2.4}$$

3. The fully renormalized pnQRPA

In [12], we showed that all these operators can be renormalized as suggested by the commutation equations

$$\begin{aligned} \left[A_{1\mu}(k), A^\dagger_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'} \left[1 - \frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \\ \left[B^\dagger_{1\mu}(k), A^\dagger_{1\mu'}(k')\right] &\approx \left[B^\dagger_{1\mu}(k), A_{1\mu'}(k')\right] \approx 0, \\ \left[B_{1\mu}(k), B^\dagger_{1\mu'}(k')\right] &\approx \delta_{k,k'}\delta_{\mu,\mu'} \left[\frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \qquad k = (I_p, I_n). \end{aligned} \tag{3.1}$$

Indeed, denoting by $C^{(1)}_{I_p,I_n}$ and $C^{(2)}_{I_p,I_n}$ the averages of the right-hand sides of (3.1) with the renormalized RPA vacuum state, the renormalized operators defined as

$$\bar{A}_{1\mu}(k) = \frac{1}{\sqrt{C^{(1)}_k}} A_{1\mu}, \qquad \bar{B}_{1\mu}(k) = \frac{1}{\sqrt{|C^{(2)}_k|}} B_{1\mu} \tag{3.2}$$

obey boson-like commutation relations

$$\begin{aligned} \left[\bar{A}_{1\mu}(k), \bar{A}^\dagger_{1\mu'}(k')\right] &= \delta_{k,k'}\delta_{\mu,\mu'}, \\ \left[\bar{B}_{1\mu}(k), \bar{B}^\dagger_{1\mu'}(k')\right] &= \delta_{k,k'}\delta_{\mu,\mu'} f_k, \qquad f_k = \mathrm{sign}\left(C^{(2)}_k\right). \end{aligned} \tag{3.3}$$

Further, these operators are used to define the phonon operator

$$C^{\dagger}_{1\mu} = \sum_{k} \big[X(k)\bar{A}^{\dagger}_{1\mu}(k) + Z(k)\bar{D}^{\dagger}_{1\mu}(k) - Y(k)\bar{A}_{1-\mu}(k)(-)^{1-\mu} - W(k)\bar{D}_{1-\mu}(k)(-)^{1-\mu}\big], \tag{3.4}$$

where $\bar{D}^{\dagger}_{1\mu}(k)$ is equal to $\bar{B}^{\dagger}_{1\mu'}(k')$ or $\bar{B}_{1\mu}(k)$ depending on whether f_k is + or −. The phonon amplitudes are determined by the equations

$$\big[H, C^{\dagger}_{1\mu}\big] = \omega C^{\dagger}_{1\mu} \qquad \big[C_{1\mu}, C^{\dagger}_{1\mu'}\big] = \delta_{\mu\mu'}. \tag{3.5}$$

Interesting properties for these equations and their solutions are discussed in our previous publications [12, 13].

4. The projected gauge of the fully renormalized pnQRPA

The renormalized ground state, i.e. the vacuum state for the phonon operator defined by the FRpnQRPA approach, is a superposition of components describing the neighboring nuclei $(N-1, Z+1)$, $(N+1, Z-1)$, $(N+1Z+1)$, $(N-1, Z-1)$. The first two components conserve the total number of nucleons $(N+Z)$ but violate the third component of isospin, T_3. By contrast, the last two components violate the total number of nucleons but preserve T_3. Actually, the last two components contribute to the violation of the ISR. However, one can construct linear combinations of the basic operators $A^{\dagger}$, A, $B^{\dagger}$, B which excite the nucleus (N, Z) to the nuclei $(N-1, Z+1)$, $(N+1, Z-1)$, $(N+1, Z+1)$, $(N-1, Z-1)$, respectively. These operators are

$$\begin{aligned}
\mathcal{A}^{\dagger}_{1\mu}(pn) &= U_pV_nA^{\dagger}_{1\mu}(pn) + U_nV_pA_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad + U_pU_nB^{\dagger}_{1\mu}(pn) - V_pV_nB_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathcal{A}_{1\mu}(pn) &= U_pV_nA_{1\mu}(pn) + U_nV_pA^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad + U_pU_nB_{1\mu}(pn) - V_pV_nB^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathbf{A}^{\dagger}_{1\mu}(pn) &= U_pU_nA^{\dagger}_{1\mu}(pn) - V_pV_nA_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad - U_pV_nB^{\dagger}_{1\mu}(pn) - V_pU_nB_{1,-\mu}(pn)(-)^{1-\mu}, \\
\mathbf{A}_{1\mu}(pn) &= U_pU_nA_{1\mu}(pn) - V_pV_nA^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu} \\
&\quad - U_pV_nB_{1\mu}(pn) - V_pU_nB^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}.
\end{aligned}$$

Indeed, in the particle representation, these operators have the expressions

$$\mathcal{A}^{\dagger}_{1\mu}(pn) = -\big[c^{\dagger}_p c_{\tilde{n}}\big]_{1\mu}, \qquad \mathcal{A}_{1\mu}(pn) = -\big[c^{\dagger}_p c_{\tilde{n}}\big]^{\dagger}_{1\mu}, \tag{4.1}$$

$$\mathbf{A}^{\dagger}_{1\mu}(pn) = \big[c^{\dagger}_p c^{\dagger}_n\big]_{1\mu}, \qquad \mathbf{A}_{1\mu}(pn) = \big[c^{\dagger}_p c^{\dagger}_n\big]^{\dagger}_{1\mu}. \tag{4.2}$$

In terms of the new operators the many-body Hamiltonian is

$$\begin{aligned}
H &= \sum_{\tau jm} E_{\tau j} a^{\dagger}_{\tau jm} a_{\tau jm} + 2\chi \sum_{pn,p'n';\mu} \sigma_{pn;p'n'} \mathcal{A}^{\dagger}_{1\mu}(pn)\mathcal{A}_{1\mu}(p'n') - X_{dp} \sum_{\substack{pn;p'\\ n';\mu}} \sigma_{pn;p'n'} \\
&\quad \times \big(\mathcal{A}^{\dagger}_{1\mu}(pn)\mathcal{A}^{\dagger}_{1,-\mu}(p'n') + \mathcal{A}_{1,-\mu}(p'n')\mathcal{A}_{1\mu}(pn)\big)(-)^{1-\mu}, \\
\sigma_{pn;p'n'} &= \frac{2}{3\hat{I}_n\hat{I}_{n'}} \langle I_p||\sigma||I_n\rangle\langle I_{p'}||\sigma||I_{n'}\rangle.
\end{aligned} \tag{4.3}$$

Here $E_{\tau I}$ denotes the quasiparticle energy. If instead of the dipole ph pairing interaction we consider the proton–neutron two-body pp interaction, then the third term of H would be

$$H_3 = -2X_{dp} \sum_{\substack{pn;p'\\ n';\mu}} \sigma_{pn;p'n'} \mathbf{A}^{\dagger}_{1\mu}(pn)\mathbf{A}_{1\mu}(p'n'). \tag{4.4}$$

Since we are interested in describing the harmonic modes which preserve the total number of nucleons, the term H_3 would not contribute at the RPA level. Indeed, this term defines a deuteron-type excitation, and consequently modifies the total number of nucleons.

The equations of motion of the operators defining the phonon operator are determined by the commutation relations

$$\left[\mathcal{A}_{1\mu}(pn), \mathcal{A}^{\dagger}_{1\mu'}(p'n')\right] \approx \delta_{\mu,\mu'}\delta_{j_p,j_{p'}}\delta_{j_n,j_{n'}} \left[U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{\hat{j}_n^2}\hat{N}_n - \frac{U_p^2 - V_p^2}{\hat{j}_p^2}\hat{N}_p\right]. \tag{4.5}$$

The average of the rhs of this equation with the PGFRpnQRPA vacuum state is denoted by

$$D_1(pn) = U_p^2 - U_n^2 + \frac{1}{2I_n+1}\left(U_n^2 - V_n^2\right)\langle\hat{N}_n\rangle - \frac{1}{2I_p+1}\left(U_p^2 - V_p^2\right)\langle\hat{N}_p\rangle. \tag{4.6}$$

The equations of motion show that the two qp energies are renormalized too:

$$E^{\mathrm{ren}}(pn) = E_p\left(U_p^2 - V_p^2\right) + E_n\left(V_n^2 - U_n^2\right). \tag{4.7}$$

The space of pn dipole states, $\mathcal{S}$, is written as a sum of three subspaces defined as

$$\begin{aligned} &\mathcal{S}_+ = \{(p,n)|D_1(pn) > 0, E^{\mathrm{ren}}(pn) > 0,\}, \quad \mathcal{S}_- = \{(p,n)|D_1(pn) < 0, E^{\mathrm{ren}}(pn) < 0,\}, \\ &\mathcal{S}_{sp} = \mathcal{S} - (\mathcal{S}_+ + \mathcal{S}_-), \\ &\mathcal{N}_\pm = \dim(\mathcal{S}_\pm), \qquad \mathcal{N}_{sp} = \dim(\mathcal{S}_{sp}), \\ &\mathcal{N} = \mathcal{N}_+ + \mathcal{N}_- + \mathcal{N}_{sp}. \end{aligned} \tag{4.8}$$

The third line of the above equations specifies the dimensions of these subspaces. In $\mathcal{S}_+$ one defines the renormalized operators

$$\bar{\mathcal{A}}^{\dagger}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}^{\dagger}_{1\mu}(pn), \qquad \bar{\mathcal{A}}_{1\mu}(pn) = \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}_{1\mu}(pn), \tag{4.9}$$

while in $\mathcal{S}_-$ the renormalized operators are

$$\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}_{1\mu}(pn), \qquad \bar{\mathcal{F}}_{1\mu}(pn) = \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}^{\dagger}_{1\mu}(pn). \tag{4.10}$$

Indeed, the operator pairs $\mathcal{A}_{1\mu}, \mathcal{A}^{\dagger}_{1\mu}$ and $\mathcal{F}_{1\mu}, \mathcal{F}^{\dagger}_{1\mu}$ satisfy commutation relations of boson type. An RPA treatment within $\mathcal{S}_{sp}$ would yield either vanishing or negative energies. The corresponding states are therefore spurious.

The equations of motion for the renormalized operators read

$$\begin{aligned} \left[H, \bar{\mathcal{A}}^{\dagger}_{1\mu}(pn)\right] = {} & E^{\mathrm{ren}}(pn)\bar{\mathcal{A}}^{\dagger}_{1\mu}(pn) + 2\chi \sum_{(p_1n_1)\in\mathcal{S}_+} \sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1\mu}(p_1n_1) \\ & - 2X_{dp} \sum_{(p_1n_1)\in\mathcal{S}_-} \sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1\mu}(p_1n_1) \\ & + 2\chi \sum_{(p_1n_1)\in\mathcal{S}_-} \sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1-\mu}(-1)^{1-\mu}(p_1n_1) \end{aligned}$$

$$
\begin{aligned}
&-2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1n_1),\\
\left[H,\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn)\right]&=|E^{\rm ren}(pn)|\bar{\mathcal{F}}^{\dagger}_{1\mu}(pn)+2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1\mu}(p_1n_1)\\
&-2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1\mu}(p_1n_1)\\
&+2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1-\mu}(-1)^{1-\mu}(p_1n_1)\\
&-2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1-\mu}(p_1n_1)(-1)^{1-\mu},\\
[H,\bar{\mathcal{A}}_{1\mu}(pn)]&=-E^{\rm ren}(pn)\bar{\mathcal{A}}_{1\mu}(pn)-2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1\mu}(p_1n_1)\\
&+2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1\mu}(p_1n_1)\\
&-2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1,-\mu}(p_1n_1)(-1)^{1-\mu}\\
&+2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1-\mu}(p_1n_1)(-1)^{1-\mu},\\
[H,\bar{\mathcal{F}}_{1\mu}(pn)]&=-|E^{\rm ren}(pn)|\bar{\mathcal{F}}_{1\mu}(pn)-2\chi\sum_{(p_1n_1)\in\mathcal{S}_-}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}_{1\mu}(p_1n_1)\\
&+2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}_{1\mu}(p_1n_1)\\
&-2\chi\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{A}}^{\dagger}_{1,-\mu}(p_1n_1)(-1)^{1-\mu}\\
&+2X_{dp}\sum_{(p_1n_1)\in\mathcal{S}_+}\sigma^{(1)}_{pn;p_1n_1}\bar{\mathcal{F}}^{\dagger}_{1-\mu}(p_1n_1)(-1)^{1-\mu},
\end{aligned}
\tag{4.11}
$$

where

$$
\sigma^{(1)}_{pn;p_1n_1}=\frac{2}{\hat{1}\hat{I}_n}\langle p||\sigma||n\rangle|D_1(pn)|^{1/2}\frac{2}{\hat{1}\hat{I}_{n_1}}\langle p_1||\sigma||n_1\rangle|D_1(p_1n_1)|^{1/2}\equiv T_{pn}T_{p_1n_1}. \tag{4.12}
$$

The phonon operator is defined as

$$
\Gamma^{\dagger}_{1\mu}=\sum_{k=(pn)}\left[X(k)\bar{\mathcal{A}}^{\dagger}_{1\mu}(k)+Z(k)\bar{\mathcal{F}}^{\dagger}_{1\mu}(k)-Y(k)\bar{\mathcal{A}}_{1-\mu}(k)(-)^{1-\mu}-W(k)\bar{\mathcal{F}}_{1-\mu}(k)(-)^{1-\mu}\right] \tag{4.13}
$$

with the amplitudes determined by the equations

$$
\left[H,\Gamma^{\dagger}_{1\mu}\right]=\omega\Gamma^{\dagger}_{1\mu},\qquad \left[\Gamma_{1\mu},\Gamma^{\dagger}_{1\mu'}\right]=\delta_{\mu,\mu'}. \tag{4.14}
$$

Thus, the phonon amplitudes are obtained by solving the PGFRpnQRPA equations

$$
\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12}\\ A_{21} & A_{22} & B_{21} & B_{22}\\ -B_{11} & -B_{12} & -A_{11} & -A_{12}\\ -B_{21} & -B_{22} & -A_{21} & -A_{22}\end{pmatrix}\begin{pmatrix} X(pn)\\ Z(pn)\\ Y(pn)\\ W(pn)\end{pmatrix}=\omega\begin{pmatrix} X(p_1n_1)\\ Z(p_1n_1)\\ Y(p_1n_1)\\ W(p_1n_1)\end{pmatrix} \tag{4.15}
$$

where the following notations have been used:

$$
\begin{aligned}
(A_{11}) &= E^{\rm ren}(pn)\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(A_{12}) &= -2X_{dp}\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{11}) &= -2X_{dp}\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{12}) &= 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(A_{21}) &= -2X_{dp}\sigma^{(1)T}_{p_1n_1;pn},\\
(A_{22}) &= |E^{\rm ren}(pn)|\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{21}) &= 2\chi\sigma^{(1)T}_{p_1n_1;pn},\\
(B_{22}) &= -2X_{dp}\sigma^{(1)T}_{p_1n_1;pn}.
\end{aligned}
\tag{4.16}
$$

Here the index T suggests the fact that the matrix is transposed. The matrix dimension for A_{11} and B_{11} is $\mathcal{N}_+ \times \mathcal{N}_+$ while for A_{22} and B_{22} it is $\mathcal{N}_- \times \mathcal{N}_-$. The off-diagonal sub-matrices A_{12} and B_{12} have the dimension $\mathcal{N}_+ \times \mathcal{N}_-$ while A_{12} and B_{12} are of the $\mathcal{N}_- \times \mathcal{N}_+$ type.

In order to solve equations (4.15) we need to know $D_1(pn)$ and, therefore, the averages of the qp's number operators, $\hat{N}_p$ and $\hat{N}_n$. These are written first in particle representation and then the particle number conserving term is expressed as a linear combination of $\mathcal{A}^\dagger\mathcal{A}$ and $\mathcal{F}^\dagger\mathcal{F}$ chosen such that their commutators with $\mathcal{A}^\dagger$, $\mathcal{A}$ and $\mathcal{F}^\dagger$, $\mathcal{F}$ are preserved. The final result is

$$
\begin{aligned}
\langle\hat{N}_p\rangle &= V_p^2(2I_p+1) + 3\big(U_p^2 - V_p^2\big)\Bigg(\sum_{\substack{n',k\\(p,n')\in\mathcal{S}_+}} D_1(p,n')(Y_k(p,n'))^2\\
&\qquad - \sum_{\substack{n',k\\(p,n')\in\mathcal{S}_-}} D_1(p,n')(W_k(p,n'))^2\Bigg),\\
\langle\hat{N}_n\rangle &= V_n^2(2I_n+1) + 3\big(U_n^2 - V_n^2\big)\Bigg(\sum_{\substack{p',k\\(p',n)\in\mathcal{S}_+}} D_1(p',n)(Y_k(p',n))^2\\
&\qquad - \sum_{\substack{p',k\\(p',n)\in\mathcal{S}_-}} D_1(p',n)(W_k(p',n))^2\Bigg).
\end{aligned}
\tag{4.17}
$$

Equations (4.15), (4.17) and (4.6) are to be simultaneously considered and solved iteratively. It is worth mentioning that using the quasiparticle representation for the basic operators $\mathcal{A}^\dagger_{1\mu}$, $\mathcal{F}^\dagger_{1\mu}$, $\mathcal{A}_{1,-\mu}(-1)^{1-\mu}$, $\mathcal{F}_{1,-\mu}(-)^{1-\mu}$, one obtains for $\Gamma^\dagger_{1\mu}$ an expression which involves the scattering pn operators. Thus, the present approach is, indeed, PGFRpnQRPA.

It is worth noting that the compatibility condition for the PGpnQRPA equations (4.15) can be written in a compact form as a dispersion equation for the excitation energies. This is explicitly given in the appendix.

5. The $2\nu\beta\beta$ process

The formalism presented above was used to describe the $2\nu\beta\beta$ process. If the energy carried by leptons in the intermediate state is approximated by the sum of the rest energy of the emitted electron and half the Q-value of the double beta decay process

$$\Delta E = m_e c^2 + \tfrac{1}{2}Q_{\beta\beta}, \tag{5.1}$$

the reciprocal value of the $2\nu\beta\beta$ half life can be factorized as

$$\left(T_{1/2}^{2\nu\beta\beta}\right)^{-1} = F\left|M_{\mathrm{GT}}\left(0_i^+ \to 0_f^+\right)\right|^2, \tag{5.2}$$

where F is an integral on the phase space, independent of the nuclear structure, while M_{GT} stands for the Gamow–Teller transition amplitude and has the expression

$$M_{\mathrm{GT}} = \sqrt{3}\sum_{k,k'} \frac{{}_i\langle 0||\beta_i^+||1_k\rangle_{ii}\langle 1_k|1_{k'}\rangle_{ff}\langle 1_{k'}||\beta_f^+||0\rangle_f}{E_k + \Delta E + E_{1^+}}. \tag{5.3}$$

In the above equation, the denominator consists of three terms: (a) ΔE, which was already defined, (b) the average value of the kth PDFRpnQRPA energy normalized to the particular value corresponding to $k = 1$, and (c) the experimental energy for the lowest 1^+ state. The indices carried by the β^+ operators indicate that they act in the space spanned by the PGFRpnQRPA states associated with the initial (i) or final (f) nucleus, respectively. The overlap m.e. of the single phonon states in the initial and final nuclei, respectively, are calculated within PGFRpnQRPA. In equation (5.3), the Rose convention for the reduced m.e. is used [25].

Note that if we restrict the pn space to $\mathcal{S}_+$ and moreover the X_{dp} interaction is missing, M_{GT} vanishes due to the second leg of the transition. Indeed, the m.e. associated with the daughter nucleus is of the type ${}_f\langle 0|(c_n^\dagger c_p)_{1\mu}(c_n^\dagger c_p)_{1\mu}|o\rangle_f$, which is equal to zero due to the Pauli principle restriction. In this case the equations of motion are of Tam–Dankoff type and therefore the ground state correlations are missing. In order to induce the necessary correlations we have either to extend the formalism in the space $\mathcal{S}_-$ or to allow the ph excitations to interact via a pairing like force. Here the two effects are simultaneously considered. Also we remark that the operator $\bar{\mathcal{A}}_{1\mu}^\dagger$ plays the role of a β^- transition operator, while when $\bar{\mathcal{F}}_{1\mu}^\dagger$ is applied on the ground state of the daughter nucleus, it induces a β^+ transition. Therefore, the $\beta\beta$ decay cannot be described by considering the β^- transition alone.

6. Numerical application

For illustration, we present the results for the transitions ^{100}Mo$\to^{100}$Ru and ^{116}Cd$\to^{116}$Sn . For these cases the energy corrections involved in equation (5.3) are

$$\begin{aligned} \Delta E(^{100}\mathrm{Mo}) &= 2.026\ \mathrm{MeV}, \qquad E_{1^+}(^{100}\mathrm{Tc}) = 0.0\ \mathrm{MeV}, \\ \Delta E(^{116}\mathrm{Cd}) &= 1.916\ \mathrm{MeV}, \qquad E_{1^+}(^{116}\mathrm{In}) = 0.0\ \mathrm{MeV}. \end{aligned} \tag{6.1}$$

The parameters defining the single particle energies are those of the spherical shell model, the deformation parameter d and the parameter k relating the quadrupole coordinate with the quadrupole bosons as shown in equation (2.2). These are fixed as described in [18]. The deformation parameter d used in this paper is the same as in [18]. The single particle parameters as well as the strengths for the two-body interactions are given in table 1 where we also present the values used in [18]. Of course, the formalisms used here and in [18] are totally different and therefore we expect that the parameters characterizing the one- and two-body interactions used in the two approaches are different.

Few comments about these data are necessary. As shown in [24] the parameter d depends linearly on the nuclear deformation, the proportionality constant being the parameter k. Comparing to the case of [18] here we used a larger deformation for ^{100}Mo and ^{100}Ru, i.e. a smaller k, and a smaller one for ^{116}Cd and ^{116}Sn. Although we use the same single particle wavefunctions due to the restrictions (4.8) as well as to the gauge conservation some single particle wavefunction become ineffective. Due to this feature, keeping for the single particle space the same dimension as in [18], the ISR would be drastically suppressed. Also

Table 1. The deformation parameter d, the pairing interaction strengths for protons (G_p) and neutrons (G_n) and the GT dipole interaction χ used in our calculations. We also give the parameter k relating the quadrupole coordinates and bosons (this is involved in the expression of the single particle energies) as well as the resulting $\log ft$ values characterizing the β^+/EC and β_- transitions of ^{100}Tc and ^{116}In, respectively. The results for $\log ft$ values, given in the right column, are compared to the experimental data from the left column.

		k		G_p (MeV)		G_n (MeV)				χ (MeV)		
	d	Here	Ref. [18]	Here	Ref. [18]	Here	Ref. [18]	ISR	$\log ft$	Here	Ref. [18]	X_{dp} (MeV)
^{100}Mo	−1.5	5.5	10.	0.18	0.28	0.288	0.26	15.995	$^{100}\mathrm{Mo} \xleftarrow{\beta^+/EC} {}^{100}\mathrm{Tc}$	0.232	0.060	1.406
									$4.45^{+0.18}_{-0.30}$ 4.65			
^{100}Ru	−0.6	5.5	3.6	0.15	0.285	0.255	0.220	12.002	$^{100}\mathrm{Tc} \xrightarrow{\beta^-} {}^{100}\mathrm{Ru}$	0.232	0.060	1.406
									4.66 4.12			
^{116}Cd	−1.8	12.	3.0	0.15	0.2	0.282	0.245	20.07	$^{116}\mathrm{Cd} \xleftarrow{\beta^+/EC} {}^{116}\mathrm{In}$	0.200	0.238	1.308
									$4.39^{+0.1}_{-0.15}$ 4.29			
^{116}Sn	−1.2	12.	2.5	0.12	0.18	0.2458	0.275	16.007	$^{116}\mathrm{In} \xrightarrow{\beta^-} {}^{116}\mathrm{Sn}$	0.200	0.238	1.308
									4.662 4.08			

Table 2. The number of single particle proton states lying above the (Z, N) core is given. The single particle space for neutrons is identical to that for protons. The dimensions of the dipole configurations which obey constraints (4.8) for mother and daughter nuclei (here) are compared with the corresponding data from [18].

	^{100}Mo		^{100}Ru		^{116}Cd		^{116}Sn	
Nucleus	Here	Ref. [18]	Here	Ref. [18]	Here	Ref. [18]	Here	Ref. [18]
The (Z, N) core	(20, 20)	(26, 26)	(20, 20)	(26, 26)	(20, 20)	(26, 26)	(20, 20)	(26, 26)
Number of states	45	20	45	20	55	27	55	27
$\dim(\mathcal{S}_+ + \mathcal{S}_-)$	138	132	141	132	189	166	186	166

the pairing contribution to the BGT strengths is also diminished. For this reason the single particle space had to be enlarged. We remark that although the dimension of the single particle space is increased by a factor of about 2, the dimension of the *pn* dipole configurations used in the pnQRPA calculations is only slightly different from that used in [18]. Of course increasing the single particle space, the pairing properties are modified. Since the protons are near to closing a major shell, the strength is to be decreased. The neutron pairing strength is a bit larger than in [18] otherwise close in magnitude.

The core system for the two decays is defined by $(Z, N) = (20, 20)$. Labeling the states according to their energies ordering, the single particle space is defined by the indices interval $[11, 55]$ and $[11, 65]$, respectively. The dimensions for the spaces $(\mathcal{S}_+, \mathcal{S}_-, \mathcal{S})$ are $(137, 1, 163)$ and $(139, 2, 175)$ for ^{100}Mo and ^{100}Ru, while for the mother and daughter nuclei of the decay ^{116}Cd$\rightarrow$^{116}Sn, they are $(189, 0, 219)$ and $(182, 4, 219)$, respectively. For both processes considered here eight iterations were necessary in order that the iteration process reaches the convergence. The dimensions of single particle and *pn* dipole configuration spaces are compared with similar data used in [18] in table 2. The strength of the dipole *pn* two-body interaction is usually taken to be

$$\chi = \frac{5.2}{A^{0.7}}\ \text{MeV}. \tag{6.2}$$

This expression was obtained by fitting the positions of the GT resonances in ^{40}Ca, ^{90}Zr and ^{208}Pb [26]. This expression provides for χ the values 0.207 and 0.187 for ^{100}Mo and ^{116}Cd, respectively. These values yield for the $\log ft$ values of the intermediate odd–odd nuclei results which deviate much from the corresponding experimental data. Moreover, even in [18] these values of the *ph* interaction strength yields half lives which deviate from the experimental data especially for ^{116}Cd. Therein, the $\log ft$ values associated with the decays of the intermediate odd–odd nuclei are well described by using χ given in table 1 and a large strength for the *pp* interaction. Indeed the ratio of the strengths for the *pp* and *ph* interactions, usually denoted by g_{pp}, is 1.6 and 1.68, respectively. For this reason we fixed χ by fitting the $\log ft$ value characterizing the β^+/EC process of the intermediate nuclei. Results for χ obtained in this way are slightly different from those provided by the expression (6.2). The parameter X_{dp} was fixed such that the $\log ft$ value characterizing the β^- decay of the intermediate odd–odd nuclei is close to the corresponding experimental data.

We note that the ISR is satisfied both for mother and daughter nuclei. In our calculation the ISR is sensitive to the dimension of the single particle basis. Indeed, choosing a basis of a smaller dimension the ISR would be underestimated. Another parameters which influence the magnitude of ISR are the pairing strengths. Indeed ISR is increasing by increasing G_n or decreasing G_p. Variation of χ and X_{dp} does not affect much ISR. However, by varying these parameters, the single beta strengths are modified according to the interaction nature. The

Table 3. The Gamow–Teller amplitude for the $2\nu\beta\beta$ decay, in units of MeV^{-1}, and the corresponding half life ($T_{1/2}$), in units of yr, are listed for two ground-to-ground transitions. The experimental half lives for the transitions of ^{100}Mo ([a][27], [b][28], [c][29], [d][30]), ^{116}Cd ([e][37]) are also given. In the second last column the results reported in [f][31] and [g][32] are given. Comparison is also made with the theoretical results from the last column reported in [h][33], [34] (unmarked) and [j][35].

	M_{GT} (MeV^{-1})	$T_{1/2}$ (yr) Present	Exp.	Reference [18]	Reference [31, 32]	Reference [33, 34, 35]
^{100}Mo→^{100}Ru	0.221	8.79×10^{18}	$(8.0 \pm 0.16) \times 10^{18}$ [a] $(0.115^{+0.03}_{-0.02}) \times 10^{20}$ [b] $0.033^{+0.02}_{-0.01} \times 10^{20}$ [c,d]	4.4×10^{18}	2.9×10^{18} [f]	1.8×10^{18} [h]
^{116}Cd→^{116}Sn	0.15998	2.02×10^{19}	$(3.2 \pm 0.3) \times 10^{19}$ [e]	3.86×10^{19}	5.1×10^{19} [g]	8.3×10^{18} 3.75×10^{19} [j]

strength is transferred to the lower energy by the attractive interaction (X_{dp}) and pushed up by the repulsive one (χ).

Using these input data we calculated the distribution of the $\beta^{\pm}$ strengths with the result shown in figure 1. The energy intervals where both distributions are large contribute significantly to the double beta transition amplitude. In plotting the β^{+} strength we ignored the values smaller than 0.01. The $\beta^{\mp}$ strengths are fragmented among the pnQRPA states, reflecting the fact that the single particle states are deformed. Note that the first peak for the β^{-} strength is the highest one while the one centered at higher energy has a large width and a fine substructure. The low energy peak is mainly determined by the attractive two-body interaction while the broad peak, i.e. the GT giant resonance, by the ph interaction. The β^{+} strength is small in magnitude and less fragmented than the β^{-} strength. Also we note that the highest energy peak is the largest one.

Calculating first the GT transition amplitude and then the Fermi integral with $G_A = 1.254$, as in [4], we obtained the results given in table 3.

From table 3, one may see that this approach provides for the half life of the double beta decay, values which are quite close to the experimental data. The results are compared with other theoretical calculations using different formalisms. Thus, in [18] a schematic Gamow–Teller proton–neutron interaction, in the particle–hole and particle–particle channels, is treated within a projected spherical single particle basis by a pnQRPA approach. The results correspond to a large value of the parameter g_{pp}. Indeed, for the two nuclei considered here, ^{100}Mo and ^{116}Cd, the parameters (χ, g_{pp}) are (0.06, 1.6) and (0.238, 1.68), respectively. In [31, 32], a realistic Bonn interaction is treated by a higher pnQRPA approach, using a single particle basis corresponding to the Woods–Saxon potential. Of course, the ISR is not obeyed given the fact that a higher pnQRPA approach is used. The results shown in the last column were obtained using for proton–neutron interaction the Paris potential in both the ph and pp channel and a pnQRPA formalism. It is worth mentioning that although within the pnQRPA approach the ISR is obeyed the large value for the pp interaction strength raises the question whether the used formalism is still valid. On the other hand, the higher pnQRPA approach, yielding the results shown in table 3 in the column labelled 'References [31, 32]', leads necessarily to a violation of the ISR. We remark that these weak points are not present in the present formalism. In [38] the standard renormalized pnQRPA was applied for calculating the rate of double beta transitions of many isotopes. Unfortunately, the list does not include the isotopes considered here and consequently we cannot compare the predictions presented here with those from the quoted reference. Another fully renormalized procedure which

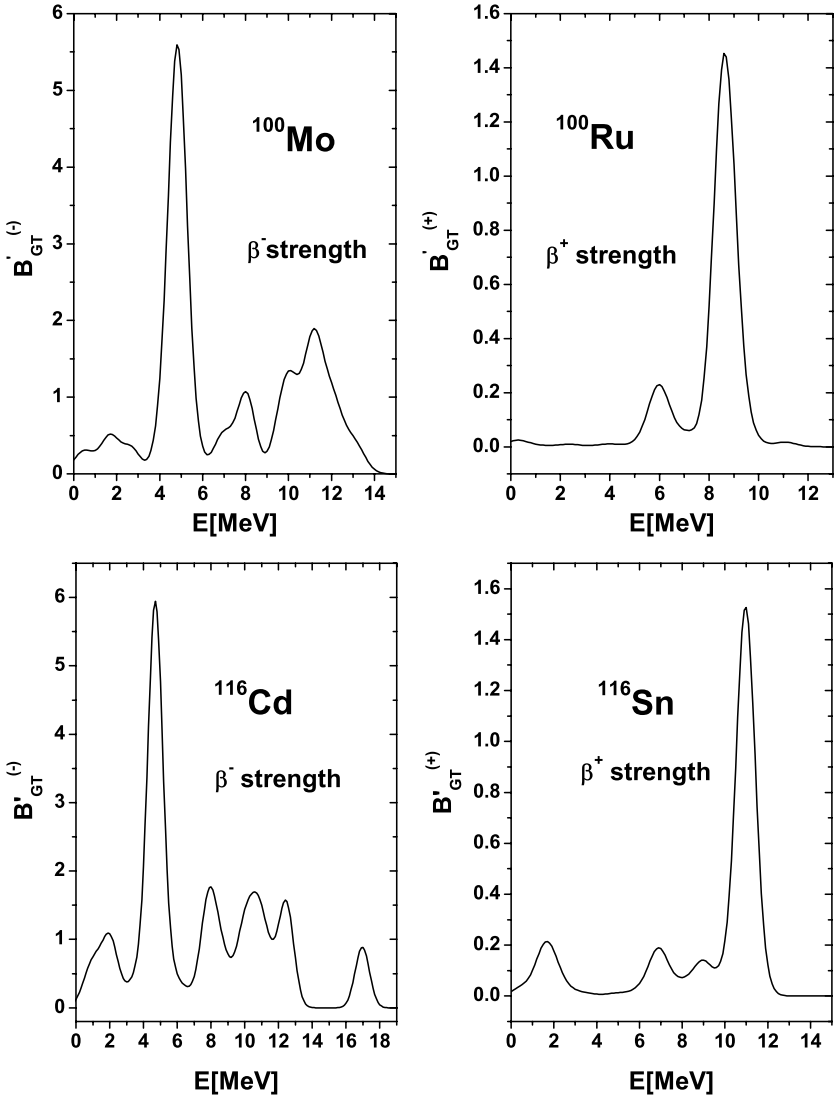

Figure 1. One-third of the single β^- strength for the mother nucleus, ^{100}Mo (upper-left panel) and ^{116}Cd (bottom-left panel), and one-third of the β^+ strength for the daughter nucleus, ^{100}Ru (upper-right panel) and ^{116}Sn (bottom-right panel), folded by a Gaussian function with a width of 1 MeV, are plotted as functions of the corresponding energies yielded by the present formalism. Note that for a given nucleus the difference $B'^{(-)}_{\rm GT} - B'^{(+)}_{\rm GT}$ is to be compared with the reduced ISR value, i.e. $N{-}Z$.

obeys the ISR was formulated in [39]. The dependence of single beta decay strengths on the particle–particle interaction was studied numerically in [40]. By contrast, in our case, the pp interaction does not contribute to the ground state correlations. Actually for this reason we replaced it by a dipole-pairing interaction. This interaction brings important contributions to the backward-going RPA amplitudes. However, even if this attractive two-body interaction is missing, the ground state correlations would not vanish due to the presence of the amplitudes $\mathcal{F}$ in the phonon operator expression. At its turn this is caused by the split of the pn dipole configurations in two orthogonal subspaces, one associated with the β^- transition while the

Table 4. The experimental summed strength for the β^- transition (second row) is compared with the corresponding theoretical values quenched by a factor 0.6, for the two considered double beta emitters.

Nucleus	^{100}Mo	^{116}Cd
$0.6\sum B(\mathrm{GT})_-$	28.96	36.2
$\sum B(\mathrm{GT})_-^{\mathrm{Exp}}$	26.69	32.7

other one to the β^+ decay. Also the ISR is obeyed. We added however the dipole-pairing interaction in order to describe the log ft values associated with the single beta transitions of the intermediate odd–odd nuclei. The eigenvalue equations and the restrictions for the averaged quasiparticle number operators, in the two renormalization approaches, are different from each other.

Another experimental result concerns the summed strength for the β^- transition, denoted, conventionally, by $\sum B_{\mathrm{GT}-}$. The experimental value of this sum covers only a fraction of the sum rule limit of $3(N-Z)$. Therefore, in order to get a fair comparison of the calculated and measured quantity, we have to quench the calculated strength by a factor 0.6 in order to account for the missing experimental strength. The results are presented in table 4.

The intermediate odd–odd nucleus, ^{100}Tc, can perform the transition β^+/EC, feeding ^{100}Mo or the β^- transition to ^{100}Ru. The same is true for ^{116}In which by means of an EC process may undergo to ^{116}Cd while through a β^- decay can feed the nucleus ^{116}Sn. The measured log ft values for these transitions are given in table 1. The corresponding theoretical results are obtained by means of the expression

$$ft_{\mp} = \frac{6160}{[{}_l\langle 1_1||\beta^{\pm}||0\rangle_l g_A]^2}, \; l = i, f. \tag{6.3}$$

In order to take account of the effect of distant states responsible for the 'missing strength' in the giant GT resonance [4] we choose $g_A = 1.0$. As we already mentioned, these single beta transitions were used as to fix the strengths of the two-body dipole–dipole interactions.

The matrix elements involved in the double beta transition amplitude of the two emitters, ^{100}Mo and ^{116}Cd, have been experimentally investigated in [37]. The first matrix elements, describing the transitions $0_i^+ \to 1^+$, were obtained from the reactions ^{100}Mo(^{3}He,t)^{100}Tc and ^{116}Cd(^{3}He,t)^{116}In, respectively, at $\theta_t \approx 0^0$, while the matrix elements for the $1^+ \to 0_f^+$ were derived from the known log ft value. For both cases the strength of the first β^- transition exhibits two bumps, one broad and called GTR1, while the second one less spread, located at lower energy and called GTR2. The centroid energies of the two resonances as well as the strength carried by each of them are compared with the theoretical results obtained with our approach, in table 5.

From the mentioned table we note that the theoretical centroid energy for GTR1 is by about 2 MeV smaller than the experimental data. Moreover, the calculated strength carried by GTR1 is smaller than the corresponding experimental data. A reason for such a discrepancy might be the relative values for the attractive and repulsive dipole–dipole interaction intensities which favors the transfer of strength from the GTR states to the states from the resonance of GTR2 and those from around 5 MeV. It is an open question whether these deviations could be washed out by a better fitting procedure or they constitute the price we have to pay for restoring the gauge symmetry. Note that the centroid for GTR2 of the ^{100}Mo$\to$ ^{100}Tc transition is fairly well described by our approach. As for the transition ^{116}Cd $\to$^{116}In, the predicted centroid of GTR2 is 1 MeV lower in energy than the corresponding experimental data. The calculated strengths for GTR2 are larger than the corresponding experimental data. It is worth

Table 5. The strengths B(GT) of the single β^- transitions from the mother nuclei to the intermediate odd–odd nuclei excited in the states of the two components, GTR1 and GTR2, of the GT giant resonance are listed. The experimental (Exp.) and theoretical (Th.) values for the centroid energies of the two resonances are also specified.

	^{100}Tc				^{116}In			
	Ex (MeV)		B(GT)		Ex (MeV)		B(GT)	
Excited states	Exp.	Th.	Exp.	Th.	Exp.	Th	Exp.	Th.
GTR1	13.3	11.16	23.1± 3.8	15.63	14.5	12.37	25.8± 4.1	18.9
GTR2	8.0	8.05	2.9±0.5	5.87	8.9	7.87	6.6±1.1	7.2

mentioning that the summed strength of the two resonances, GTR1 and GTR2, is reasonably close to the corresponding experimental data. A specific feature for our formalism is that states around the GTR1 centroid contribute to the peak seen in the strength distribution for the single β^+ decay of the daughter nuclei. Also for both nuclei it seems that the resonance around 5 MeV for the β^- decay does not contribute at all to the double beta decay. Indeed, in this region, the matrix element for the second leg of transition is almost vanishing. In this respect we note that in the case of ^{116}Cd, a bunch of states below 2 MeV contribute to the double beta decay rate.

Before closing few statements clarifying the objective of this paper is necessary. From table 3 we see that this paper provides a half life for ^{100}Mo which agrees with the data from [27, 28] better than the result of [18]. Moreover the agreement with the data for the decay of ^{116}Cd is of the same quality as for that reported in the quoted reference. However, the aim of this paper is not that of improving the description agreement with the data. Indeed, the nice agreement of [18] was obtained with large value for the g_{pp} ratio. Indeed, for small g_{pp} the amplitude M_{GT} is not affected at all and thus the pp interaction could be ignored. The effect of lowering the M_{GT} values is recorded for g_{pp} close or larger than unity, depending on the nucleus considered. Results of [18] correspond to the values 1.6 and 1.68 for ^{100}Mo and ^{116}Cd, respectively. These values lie close to the pnQRPA breaking point and therefore one cannot say how reliable is the approach used. To conclude, irrespective of the single particle basis and the two-body interaction, the pp channel is not needed in the region of small g_{pp} (less than the breaking point value by 0.15) and is decreasing dramatically the magnitude of the matrix element defining the second leg of the double beta process when the g_{pp} is approaching the pnQRPA breaking point value. This was in fact the motivation for higher pnQRPA approaches [9–12]. The drawback of the higher pnQRPA approaches is that the ISR is violated by an amount of 20–30%. Actually, this is the context of our approach. Indeed as shown before our formalism is a fully renormalized pnQRPA formalism and moreover preserves the total number of nucleons, i.e., is invariant to the gauge transformation which results in satisfying ISR. We recall that the sum rule represents a consistency test of the approximations used for the many-body systems. The gauge invariance condition infers that the pp interaction does not contribute at all in this approach. However, we need an attractive interaction responsible for the pnQRPA correlations. Here this interaction was chosen to be the dipole-pairing interaction whose strength is X_{dp}. This is not considered in addition to the pp interaction but is taken alone. The pp interaction has been commonly used for double beta decay since Cha [41] noticed that the strength of the single β^+ decay is very sensitive to the variation of the pp interaction strength. Its strength, instead of being derived from the ph interaction strength using the Pandya transformation, was always considered to be a free parameter. By contrast, the dipole-pairing interaction cannot be connected to the ph interaction by a simple

recoupling transformation. Indeed, when applied on the ground state of a (Z, N) nucleus, such an interaction may lead to a state describing either a nucleus $(Z + 2, N - 2)$ or one of the type $(Z - 2, N + 2)$. Also while in the standard pnQRPA calculations the position of the centroid of the GT giant resonance is insensitive to the magnitude of the pp interaction, here the X_{dp} interaction does influence the location of this resonance.

Therefore, this interaction may bridge the mother and daughter nuclei in a double β^- or a double β^+ process.

We note that renormalization procedures include in the mutual commutators of the basic dipole operators $A(pn)_{1\mu}$, $A^\dagger(p'n')_{1\mu'}$, besides the unity number, the monopole part of the exact result. However, the additional term is equated with its average value on the renormalized pnQRPA vacuum. In this way the Pauli principle is partially restored with respect to the standard pnQRPA description.

We may say that it is the first time when numerical calculations are performed for a fully renormalized pnQRPA and a gauge symmetry restored. Generally speaking, whenever some beauty conditions, like fully renormalization and gauge symmetry restoration, are met a certain tribute is expected to be payed. Thus there are some specific weak points which require further improvements. Indeed, the average of the quasiparticle number operators has been approximately calculated. We feel that a better expression can be found for this quantity which is essential for the adopted iterative procedure. We hope that a better representation for the average number of quasiparticles will speed up the convergence of the iterative process. Moreover, this will allow us to extend our calculations to heavy nuclei. The renormalized vacuum state is characterized by a non-vanishing average number of quasiparticles. That means that the pnQRPA features are determined by the pairing properties not only through the occupation probabilities U^2 and V^2 but also by the averages of quasiparticle number operators. The question which arises is whether the pnQRPA may influence the pairing properties. A positive answer could supply us with a unifying variational principle for both vacua of quasiparticle and pnQRPA boson, respectively. This goal was in fact touched within a different context by Jolos *et al* [42]. These features concerning the description of the quasiparticle number operators in a better way as well as describing the BCS and the pnQRPA in a unified fashion, by a set of coupled equations derived from a unique variational principle, will be implemented in a subsequent paper.

7. Conclusions

Summarizing the results of this paper, one may say that restoring the gauge symmetry from the fully renormalized pnQRPA, one obtains a realistic description of the transition rate and moreover the ISR is obeyed. As shown in this paper, it seems that there is no need to include the pp interaction in the many-body treatment of the process. Indeed, in the framework of a FRpnQRPA approach this interaction violates the total number of particle and consequently the gauge projection process makes it ineffective. For this reason the pp interaction was not included in the model Hamiltonian.

Note that the hypothesis saying that the double beta process consists of two successive single β^- decays requires a consistent description of the double beta and single beta processes. In our formalism, actually, this feature is met since the rate of the double beta decay and the $\log ft$ values associated with the single beta decays of the intermediate odd–odd nuclei are realistically described. Another issue which is worth mentioning refers to the chain of approximations of the many-body Hamiltonian. A measure of the consistency of all these approximations is the ISR which in our case is satisfied to a high accuracy.

The attractive interaction of ph dipole-pairing type is responsible for the ground state correlations. To a less extent these are also caused by the $\mathcal{F}$ components of the new phonon operator. The projection of gauge is essential for restoring the ISR. The gauge projection of the pnQRPA was previously achieved in [36] where the ISR is anyway satisfied within the unprojected picture. By contrast therein the effect of projection is small.

The GPFRpnQRPA equations consist of FRpnQRPA equation (5.12) supplemented by equations (5.6) and (5.17) which must be simultaneously solved by an iteration procedure. Since the two-body interaction is a separable interaction, equation (5.2) may be replaced by the dispersion equation (A.1) for energies and equations (A.3) for the four phonon amplitudes. An extensive study of all existent data, with the formalism described in this paper, will be presented in a subsequent work.

Acknowledgments

This work was supported by the Romanian Ministry for Education Research Youth and Sport through the CNCSIS project ID-1038/2008.

Appendix

The compatibility condition for the homogeneous system of equations determining the phonon amplitudes yields two dispersion equations for ω:

$$4\left(\chi^2 - X_{dp}^2\right)\left[\mathcal{R}_-^+ - \mathcal{R}_+^-\right]\left[\mathcal{R}_+^+ - \mathcal{R}_-^-\right] + 4\chi\left[\sum_{\mathcal{S}_+}\frac{T_{pn}^2 E^{\mathrm{ren}}(pn)}{\omega^2 - (E^{\mathrm{ren}}(pn))^2} + \sum_{\mathcal{S}_-}\frac{T_{pn}^2|E^{\mathrm{ren}}(pn)|}{\omega^2 - |E^{\mathrm{ren}}(pn)|^2}\right] = 1, \tag{A.1}$$

with

$$\mathcal{R}_\pm^+ = \sum_{(p_1n_1)\in\mathcal{S}_+}\frac{T_{p_1n_1}^2}{\omega \pm E^{\mathrm{ren}}(p_1n_1)}, \qquad \mathcal{R}_\pm^- = \sum_{(p_1n_1)\in\mathcal{S}_-}\frac{T_{p_1n_1}^2}{\omega \pm |E^{\mathrm{ren}}(p_1n_1)|}. \tag{A.2}$$

The phonon amplitudes can be analytically determined. Indeed, the GPFRpnQRPA equations yield the following expressions for the four amplitudes:

$$\begin{aligned} X(pn) &= 2\frac{T_{pn}^2}{\omega - E^{\mathrm{ren}}(pn)}(\chi\mathcal{X} - X_{dp}\mathcal{Y}), \quad W(pn) = -2\frac{T_{pn}^2}{\omega + |E^{\mathrm{ren}}(pn)|}(\chi\mathcal{X} - X_{dp}\mathcal{Y}), \\ Z(pn) &= 2\chi\frac{T_{pn}^2}{\omega - |E^{\mathrm{ren}}(pn)|}(\chi\mathcal{Y} - X_{dp}\mathcal{X}), \quad Y(pn) = -2\chi\frac{T_{pn}^2}{\omega + E^{\mathrm{ren}}(pn)}(\chi\mathcal{Y} - X_{dp}\mathcal{X}). \end{aligned} \tag{A.3}$$

The constant factors $\mathcal{X}$ and $\mathcal{Y}$ have the expressions

$$\begin{aligned} \mathcal{X} &= \sum_{\mathcal{S}_+} T_{p_1n_1}X(p_1n_1) + \sum_{\mathcal{S}_-} T_{p_1n_1}W(p_1n_1), \\ \mathcal{Y} &= \sum_{\mathcal{S}_+} T_{p_1n_1}Y(p_1n_1) + \sum_{\mathcal{S}_-} T_{p_1n_1}Z(p_1n_1). \end{aligned} \tag{A.4}$$

The two factors are related by

$$\mathcal{Y} = \frac{1}{X_{dp}}\left[\chi - \frac{1}{2\left(\mathcal{R}_-^+ - \mathcal{R}_+^-\right)}\right]\mathcal{X} \equiv \mathcal{U}\mathcal{X}. \tag{A.5}$$

Finally, the independent constant factor $\mathcal{X}$ is determined from the normalization condition of the phonon operator. The result is

$$\mathcal{X}^{-2} = 4\left[\sum_{\mathcal{S}_+}\frac{T_{pn}^2}{(\omega - E^{\rm ren}(pn))^2} - \sum_{\mathcal{S}_-}\frac{T_{pn}^2}{(\omega + |E^{\rm ren}(pn)|)^2}\right](\chi - X_{dp}\mathcal{U})^2$$
$$+4\left[\sum_{\mathcal{S}_-}\frac{T_{pn}^2}{(\omega - |E^{\rm ren}(pn)|)^2} - \sum_{\mathcal{S}_+}\frac{T_{pn}^2}{(\omega + E^{\rm ren}(pn))^2}\right](X_{dp} - \chi\mathcal{U})^2. \tag{A.6}$$

Having $\mathcal{X}$ and $\mathcal{Y}$ determined, the phonon amplitudes are readily obtained by means of equations (A.3).

References

[1] Primakof H and Rosen S 1959 *Rep. Prog. Phys.* **22** 125
[2] Haxton W C and Stephenson G J Jr 1984 *Prog. Part. Nucl. Phys.* **12** 409
[3] Vergados J D 2001 *Phys. Rep.* **361** 1
[4] Suhonen J and Civitarese O 1998 *Phys. Rep.* **300** 123
[5] Tomoda T 1991 *Rep. Prog. Phys.* **54** 53
[6] Faessler A 1988 *Prog. Part. Nucl. Phys.* **21** 183
[7] Raduta A A 2002 *Prog. Part. Nucl. Phys.* **48** 233
[8] Vogel P *et al* 1986 *Phys. Rev. Lett.* **57** 3148
[9] Raduta A A, Faessler A and Stoica S 1991 *Nucl. Phys.* A **534** 149
[10] Raduta A A, Faessler A, Stoica S and Kaminsky W 1991 *Phys. Lett.* **254** 7
[11] Toivanen J and Suhonen J 1995 *Phys. Rev. Lett.* **75** 410
[12] Raduta A A, Raduta C M, Kaminski W and Faessler A 1998 *Nucl. Phys.* A **634** 497
[13] Raduta A A, Raduta C M and Codirla B 2000 *Nucl. Phys.* A **678** 382
[14] Raduta A A, Simkovich F and Faessler A 2000 *J. Phys. G: Nucl. Part. Phys.* **26** 793
[15] Raduta C M and Raduta A A 2005 *Nucl. Phys.* A **756** 153
[16] Raduta A A, Delion D S and Lo Iudice N 1993 *Nucl. Phys.* A **564** 185
[17] Raduta A A, Escuderos A, Faessler A, Moya de Guerra E and Sarriguren P 2004 *Phys. Rev.* C **69** 064321
[18] Raduta A A, Raduta C M and Escuderos A 2005 *Phys. Rev.* C **71** 024307
[19] Raduta A A, Lo Iudice N and Ursu I I 1995 *Nucl. Phys.* A **584** 84
[20] Raduta A A, Escuderos A and Moya de Guerra E 2002 *Phys. Rev.* C **65** 024312
[21] Raduta A A, Faessler A and Delion D S 1993 *Nucl. Phys.* A **564** 185
Raduta A A, Faessler A and Delion D S 1993 *Phys. Lett.* B **312** 13
[22] Raduta A A, Delion D S and Faessler A 1997 *Nucl. Phys.* A **617** 176
[23] Nilsson S G 1955 *Mat. Fys. Medd. K. Dan. Vid. Selsk.* **29** 1
[24] Raduta A A, Ceausescu V, Gheorghe A and Dreizler R M 1982 *Nucl. Phys.* A **381** 253
[25] Rose M E 1957 *Elementary Theory of Angular Momentum* (New York: Wiley)
[26] Homma H *et al* 1996 *Phys. Rev.* C **54** 2972
[27] Elliot S R and Vogel P 2002 *Annu. Rev. Nucl. Part. Sci.* **52** 115
[28] Ejiri H *et al* 1991 *Phys. Lett.* B **258** 17
[29] Kirsten T *et al* 1986 *Proc. Int. Symp. on Nuclear Beta Decays and Neutrino* ed T Kotani, H Ejiri and E Takasugi (Singapore: World Scientific) p 81
[30] Vasil'ev S I *et al* 1990 *JETP Lett.* **51** 622
[31] Suhonen J and Civitarese O 1994 *Phys. Rev.* C **49** 3055
[32] Aunola M and Suhonen J 1996 *Nucl. Phys.* A **602** 133
[33] Klapdor H V and Grotz K 1984 *Phys. Lett.* B **142** 323
[34] Grotz K and Klapdor H V 1985 *Phys. Lett.* B **157** 242
[35] Hirsch M, Wu X R, Klapdor-Kleingrothaus H V, Ching C-R and Ho T-H 1994 *Phys. Rep.* **242** 403
[36] Civitarese O *et al* 1991 *Nucl. Phys.* A **524** 404

[37] Akimune H *et al* 1997 *Phys. Lett.* B **394** 23
[38] Toivanen J and Suhonen J 1997 *Phys. Rev.* C **55** 2314
[39] Rodin V and Faessler A 2002 *Phys. Rev.* C **66** 051393
[40] Pacearescu L, Rodin V, Simkovic F and Faessler A 2003 *Phys. Rev.* C **68** 064310
[41] Cha D 1983 *Phys. Rev.* C **27** 2269
[42] Jolos R V and Rybarska-Nawrocka W 1980 *Z. Phys.* A **296** 73